

On the Stochastic Limit of Quantum Field Theory

L. Accardi, J.Gough and Y.G. Lu

Abstract

The weak coupling limit for a quantum system, with discrete energy spectrum, coupled to a bose reservoir with the most general linear interaction is considered: under this limit we have a quantum noise processes substituting for the field. We obtain a limiting evolution unitary on the system and noise space which, when reduced to the sytem's degrees of freedom, provide the master and Langevin equations that are postulated on heuristic grounds by physicists. In addition we give a concrete application of our results by deriving the evolution of an atomic system interacting with the electrodynamic field without recourse to either rotating wave or dipole approximations.

1 QUANTUM THEORY OF DAMPING

Irreversible quantum evolutions now play a fundamental role in many areas of physics especially quantum optics. A large body of physical literature has been built up around the problem of describing in a stochastic model the effect of a source of quantum noise on a given quantum mechanical system, emphasising the quantum stochastic properties of the source of the quantum noise. However, quite generally, the classical approach to the stochasticity which is taken ignores the essential quantum nature of the problem. In this paper we wish to develop the weak coupling approach [1] to quantum damping. In a sense the noise feilds used to model physical sources are still quantum in nature, as will be explained later, and because they are arrive at from a well defined physical scaling limit do not require us to put in the desired features of the noise by hand.

The derivations of the quantum master equations and quantum Langevin equations for open quantum systems which are present in the current physics literature are well motivated from the physical point of view, cf. [2], however mathematically imprecise. The heuristic procedure to render the reservoir, to which the system is coupled, into a source of quantum noise via some *markovian approximation* is generally ill-defined and reliant on second order perturbation theory.

On the other hand, the weak coupling limit for an open quantum system gives a device for obtaining irreversible evolutions. Mathematically rigorous derivations of the Langevin and master equations along these lines have been given for certain specific models by Pulé [3] and Davies [4]. Other attempts have been made to approximate a quantum reservoir by fitting Quantum Brownian Motions (QBM's), cf. [5],[6] and [7].

In a long series of papers [[1],[8],[9],[10],[11]], Accardi, Frigerio and Lu have developed their approach whereby a quantum reservoir can be reduced to a quantum stochastic noise source via a weak coupling limit procedure. The theory is mathematically rigorous while at the same time applicable to the wide range of phenomena considered by physicists and gives a precise description of the reservoir as a quantum noise source. The main mathematical device for establishing convergence of the reservoir feilds is a quantum central limit theorem; that is a central limit theorem for non-commuting observables: the theory of non-commutative probability, or *Quantum Probability* as it is more correctly known, affords the necessary mathematical framework to interpret the limit process. Indeed the Fock space descriptions of a bose reservoir and a quantum stochastic processes are so similar in nature, it is not surprising, with hindsight, that the former may be reduced to the latter following some limiting procedure.

Our objective in this paper is to review the results of the programme of Accardi, Frigerio and Lu so far and to extend their theory so as to deal with the most general interaction between a system and a noise source encountered in physical theories. This we do and show that the energy shifts, linewidths, master equations and Langevin equations, arising for the system as a result of its coupling to the noise field, concur with those obtain by earlier researchers [2]. However we have, in addition, also a quantum stochastic description of the reservoir noise fields themselves. As a concrete application of our theory we consider the particular case where a quantum electrodynamic field acts as reservoir, however we stress that this is only one of the many applications of our theory.

We shall discuss only minimal coupling interactions, that is interactions linear in the creation/ annihilation operators for the reservoir. In a later paper we discuss how to treat the situations where the interaction is of polynomial type.

1.1 Open Quantum Systems

We consider a system (S) coupled to a reservoir (R). The system (S) is to be a quantum mechanical: its state space shall be a separable Hilbert space \mathcal{H}_S . The reservoir, on the other hand, is comprised of one or indeed several quantum fields, and so has infinitely many degrees of freedom. We shall consider a bosonic reservoir; the state space for (R) is the bosonic Fock space \mathcal{H}_R over a separable Hilbert space \mathcal{H}_R^1 ; in standard notation we write $\mathcal{H}_R = \Gamma_B(\mathcal{H}_R^1)$. \mathcal{H}_R^1 is again to be a separable Hilbert space and may quite generally describe not only one but several individual species of particle in the reservoir. For instance, consider several species of particle P_1, P_2, P_3, \dots in the reservoir and suppose that $\mathcal{H}_R^1 = \oplus_j \mathcal{H}_{P_j}^1$, where $\mathcal{H}_{P_j}^1$ is the state space for particle type P_j , then

$$\mathcal{H}_R = \Gamma_B(\mathcal{H}_R^1) = \Gamma_B(\oplus_j \mathcal{H}_{P_j}^1) = \otimes_j \Gamma_B(\mathcal{H}_{P_j}^1). \quad (1)$$

The space \mathcal{H}_R^1 is referred to as the (combined) one particle state space for the reservoir. The overall state space for the combined system and reservoir is $\mathcal{H}_S \otimes \mathcal{H}_R$. The vacuum vector of the reservoir space will be denoted throughout as Ψ_R . In the following we consider only bosonic species in the reservoir, however it is also possible to work with fermions [9]. The dynamics of the combined system and reservoir is governed by the formal Hamiltonian $H^{(\lambda)}$ which we may write as

$$H^{(\lambda)} = H^{(0)} + \lambda H_I; \quad (2)$$

that is, as the sum of a free Hamiltonian $H^{(0)}$ and an interaction H_I , with λ a real coupling parameter. $H^{(0)}$ is to be expressible as

$$H^{(0)} = H_S \otimes 1_R + 1_S \otimes H_R, \quad (3)$$

where H_S and H_R are self-adjoint operators on the spaces \mathcal{H}_S and \mathcal{H}_R respectively. For each λ , we consider the unitary operator $V_t^{(\lambda)}$ on $\mathcal{H}_S \otimes \mathcal{H}_R$ defined by

$$V_t^{(\lambda)} = \exp \frac{t}{i\hbar} H^{(\lambda)}. \quad (4)$$

This gives the time evolution under $H^{(\lambda)}$. A standard device in perturbation theory is to transform to the interaction picture; this involves introducing the operator

$$U_t^{(\lambda)} = V_t^{(0)\dagger} V_t^{(\lambda)}. \quad (5)$$

$U_t^{(\lambda)}$ is a unitary operator on $\mathcal{H}_S \otimes \mathcal{H}_R$ called the wave operator at time t . We note that $\{U_t^{(\lambda)} : t \in \mathbb{R}\}$ is a left $v_t^{(0)}$ -cocycle, that is it satisfies the relation

$$U_{t+s}^{(\lambda)} = v_t^{(0)}(U_s^{(\lambda)}) U_t^{(\lambda)}. \quad (6)$$

We define the time-evolutes for $X \in \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_R)$ as

$$\begin{aligned} v_t^{(\lambda)}(X) &= V_t^{(\lambda)\dagger} X V_t^{(\lambda)}, \\ u_t^{(\lambda)}(X) &= U_t^{(\lambda)\dagger} X U_t^{(\lambda)}. \end{aligned} \quad (7)$$

The Schrödinger equation for the time evolutions are

$$\frac{\partial}{\partial t} V_t^{(\lambda)} = \frac{1}{i\hbar} H^{(\lambda)} V_t^{(\lambda)}$$

and

$$\frac{\partial}{\partial t} U_t^{(\lambda)} = \frac{\lambda}{i\hbar} v_t^{(0)}(H_I) U_t^{(\lambda)}. \quad (8)$$

With these we deduce the associated Heisenberg equations

$$\frac{\partial}{\partial t} v_t^{(\lambda)}(X) = \frac{1}{i\hbar} [v_t^{(\lambda)}(X), v_t^{(\lambda)}(H^{(\lambda)})] = \frac{1}{i\hbar} v_t^{(\lambda)}([X, H^{(\lambda)}])$$

and

$$\frac{\partial}{\partial t} u_t^{(\lambda)}(X) = \frac{\lambda}{i\hbar} u_t^{(\lambda)}([X, v_t^{(0)}(H_I)]). \quad (9)$$

From (8) we obtain the integral equation

$$U_t^{(\lambda)} = 1 + \frac{\lambda}{i\hbar} \int_0^t ds v_s^{(0)}(H_I) U_s^{(\lambda)}, \quad (10)$$

and consequently the iterated Born series

$$U_t^{(\lambda)} = 1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n v_{t_1}^{(0)}(H_I) \dots v_{t_n}^{(0)}(H_I) \right). \quad (11)$$

This may be resummed as

$$U_t^{(\lambda)} = \mathcal{T} \exp \left\{ \frac{\lambda}{i\hbar} \int_0^t ds v_s^{(0)}(H_I) \right\}. \quad (12)$$

where \mathcal{T} denotes time ordering.

Before we continue, we must say more about how to interpret the formal sum of $H^{(0)}$ and H_I . Firstly we assume that $H^{(0)}$ and H_I are self-adjoint operators on $\mathcal{H}_S \otimes \mathcal{H}_R$. We shall assume that, for sufficiently small λ and bounded t , the iterated series (11) is uniformly convergent and is bounded on $\mathcal{H}_S \underline{\otimes} \mathcal{E}(\mathcal{H}_R^1)$, the algebraic tensor product of \mathcal{H}_S and the set of exponential vectors. From the cocycle relation (1.1:6) we have that if we define the unitary operator $V_t^{(\lambda)}$ by

$$V_t^{(\lambda)} = V_t^{(0)} U_t^{(\lambda)} \quad (13)$$

then $\{V_t^{(\lambda)} : t \in \mathbb{R}\}$ gives a strongly continuous unitary group whose generator $\frac{1}{\hbar} H^{(\lambda)}$ is formally given as $H^{(\lambda)} = H^{(0)} + \lambda H_I$. The time evolution in the Heisenberg picture is then given by

$$v_t^{(\lambda)}(X) = u_t^{(\lambda)}(v_t^{(0)}(X)). \quad (14)$$

1.2 The Free Evolution

For $\mathcal{H}_R^1 = L^2(\mathbb{R}^3)$ (momentum space), we have creation/ annihilation operators $a^\sharp(\mathbf{k})$ satisfying

$$[a(\mathbf{k}), a(\mathbf{k}')^\dagger] = \delta(\mathbf{k} - \mathbf{k}'), [a(\mathbf{k})^\dagger, a^\dagger(\mathbf{k}')] = 0, [a(\mathbf{k}), a(\mathbf{k}')] = 0. \quad (15)$$

The creation/ annihilation fields on \mathcal{H}_R are defined, for $g \in \mathcal{H}_R^1$, by

$$A^\dagger(g) = \int d^3k g(\mathbf{k}) a^\dagger(\mathbf{k}), A(g) = \int d^3k \bar{g}(\mathbf{k}) a(\mathbf{k}). \quad (16)$$

From (16) we obtain the canonical commutation relations (CCR):

$[A(h), A^\dagger(f)] = \langle h, f \rangle$, while $[A(h), A(f)] = 0$. We take H_R to be the second quantization of an operator H_R^1 on \mathcal{H}_R^1 given by

$$(H_R^1 f)(k) = \hbar \omega(k) f(k). \quad (17)$$

H_R may then be expressed as

$$H_R = \int dk \hbar \omega(k) a^\dagger(k) a(k). \quad (18)$$

Note that $v_t^{(0)}(1_S \otimes A^\sharp(g)) = 1_S \otimes A^\sharp(S_t g)$; where we have introduced the unitary operator S_t on \mathcal{H}_R^1 given by

$$S_t = \exp \left\{ -\frac{t}{i\hbar} H_R^1 \right\}; \quad (19)$$

that is $(S_t f)(k) = e^{i\omega(k)t} f(k)$.

1.3 The standard approach to the quantum Langevin equation

Consider an interaction of the type

$$H_I = i\hbar\{D \otimes A^\dagger(g) - D^\dagger \otimes A(g)\}. \quad (20)$$

where $D \in \mathcal{B}(\mathcal{H}_S)$ has a harmonic free-evolution; that is $\frac{1}{i\hbar}[D, H_S] = -i\omega D$. Then we have

$$v_t^{(0)}(H_I) = i\hbar\{D \otimes A^\dagger(S_t^\omega g) - D^\dagger \otimes A(S_t^\omega g)\}, \quad (21)$$

where

$$S_t^\omega = e^{-i\omega t} S_t, \quad (22)$$

that is, $(S_t^\omega f)(\mathbf{k}) = e^{i(\omega(\mathbf{k}) - \omega)t} f(\mathbf{k})$. Now write $X_t = u_t^{(\lambda)}(X \otimes 1_R)$, for $X \in \mathcal{B}(\mathcal{H}_S)$, then from (9) we have

$$\frac{\partial X_t}{\partial t} = \lambda\{A_t^\dagger(S_t^\omega g)[X, D]_t - [X, D^\dagger]_t A_t(S_t^\omega g)\}, \quad (23)$$

where $A_t^\#(f) = u_t^{(\lambda)}(1_S \otimes A^\#(f))$.

Now one can show that $A_t(f) = \lambda \int_0^\infty ds D_s \langle f, S_s^\omega g \rangle + 1_S \otimes A(f)$, so substituting back into (23) gives

$$\frac{\partial X_t}{\partial t} = \lambda^2 \int_0^\infty \{D_s^\dagger \phi^\omega(s-t)[X, D]_t - [X, D^\dagger]_t \phi^\omega(t-s)D_s\} + \lambda\{\xi_t^\omega[X, D]_t - [X, D^\dagger]_t \xi_t^{\omega\dagger}\}, \quad (24)$$

where $\phi^\omega(t) = \langle g, S_{-t}^\omega g \rangle = \int d^3k |g(\mathbf{k})|^2 e^{-i(\omega(\mathbf{k}) - \omega)t}$ and $\xi^\omega(t) = 1_S \otimes A(S_t^\omega g)$. In standard terminology $\phi(t)$ is called the memory function and ξ_t^ω the fluctuating quantum force [2,17] or input field [18], albeit in the interaction picture. One notes that, in the vacuum state, ξ_t^ω is gaussian distributed and all first and second moments vanish except the two-point function

$$\langle \Psi_R, \xi_t^\omega \xi_s^{\omega\dagger} \Psi_R \rangle = \langle S_t^\omega g, S_s^\omega g \rangle = \phi^\omega(t-s). \quad (25)$$

The standard approach taken at this juncture is to introduce the so-called *first Markov approximation*. Here, for example, one takes $\mathcal{H}_R^1 = L^2(\mathbb{R})$, $g = \sqrt{\frac{\kappa}{2\pi}}$ (constant) and $\omega(k) = k$. Then

$$\phi^\omega(t) = \int_{-\infty}^\infty dk \frac{\kappa}{2\pi} e^{i(k+\omega)t} = \kappa e^{i\omega t} \delta(t). \quad (26)$$

There are, however, several important objections to be made to this approach. Firstly any physical details specific to the reservoir must be put in by hand. Secondly, the spectrum of H_R^1 is here unbounded below, this is necessary to produce the delta function correlation of white noise. From a physical point of view this is unacceptable as H_R^1 must be bounded below for stability. Finally, the fact that the frequency spectrum $\omega(k) = k$ is unbounded below precludes any possibility of dropping the rotating wave approximation.

1.4 The Weak Coupling Limit

We now describe the ideas behind the weak coupling limit in the simplest situation where we have taken a dipole and rotating wave approximation. We define the following reservoir operator

$$B_t^{(\omega, \lambda)}(g) = A(\lambda \int_0^t dt_1 S_{t_1}^\omega g) = \lambda \int_M \mu(dk) \int_0^t dt_1 e^{-i(\omega(k) - \omega)t_1} \bar{g}(k) a(k). \quad (27)$$

Calculating the two-point vacuum expectations gives

$$\begin{aligned} \langle \Psi_R(0), B_t^{(\omega, \lambda)}(g) B_s^{(\omega, \lambda)\dagger}(f) \Psi_R(0) \rangle &= \lambda^2 \int_0^t dt_1 \int_0^s ds_1 \langle S_{t_1}^\omega g, S_{s_1}^\omega f \rangle \\ &\equiv \int_0^{\lambda^2 t} du \int_{u/\lambda^2 - s}^{u/\lambda^2} d\tau \langle S_\tau^\omega g, f \rangle, \end{aligned} \quad (28)$$

where we have substituted $u = \lambda^2 t_1$ and $\tau = t_1 - s_1$.

In order to obtain a non-trivial two-point function in the limit $\lambda \rightarrow 0$ we must rescale time as

$$t \hookrightarrow t/\lambda^2; \quad (29)$$

this is known as the van Hove or weak coupling limit in physics. One finds

$$\lim_{\lambda \rightarrow 0} \langle \Psi_R, B_{t/\lambda^2}^{(\omega, \lambda)}(g) B_{s/\lambda^2}^{(\omega, \lambda)\dagger}(f) \Psi_R \rangle = \min(t, s) \int_{-\infty}^{\infty} d\tau \langle S_{\tau}^{\omega} g, f \rangle. \quad (30)$$

Physically the limit $\lambda \rightarrow 0$ with $t \hookrightarrow t/\lambda^2$ allows us to consider progressively weaker interactions which are allowed to run over increasingly larger periods of time and so we obtain the long term cumulative effect of the interaction on the system. Now the creation and annihilation operators are gaussian in the vacuum state and, as a result, so too are the operators $B_{t/\lambda^2}^{(\omega, \lambda)\dagger}$.

Furthermore the limiting two-point function (29) is suggestive of the correlation function of a Brownian motion. However, an interpretation of the above in terms of classical Brownian motion is erroneous as it ignores the essentially quantum probabilistic nature of these processes.

1.5 The Interaction

For technical reasons we work with a system Hamiltonian H_S which has discrete spectrum. H_R^1 is taken to be bounded below as required from physics. The type of interaction H_I which we wish to study is of the form

$$H_I = i\hbar \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} \{ D_j^{\omega} \otimes A^{\dagger}(g_j^{\omega}) - D_j^{\omega\dagger} \otimes A(g_j^{\omega}) \}, \quad (31)$$

where F is a discrete subset of \mathbb{R} . For each $\omega \in F$, we take $D_j^{\omega} \in \mathcal{B}(\mathcal{H}_S)$ to have harmonic free evolution with frequency ω :

$$\frac{1}{i\hbar} [D_j^{\omega}, H_S] = -i\omega D_j^{\omega}, \quad j = 1, \dots, N(\omega). \quad (32)$$

Thus the superscript ω labels harmonic frequency and $j = 1, \dots, N(\omega)$ the degeneracy of that frequency. An interaction similar to (30) has been treated in [10], however there the test functions g_j^{ω} were taken to be equal for each value of ω . Our reasons for studying (30) above are because it allows us to treat the most general interactions encountered in physics. Typically in quantum field theory one considers an interaction of the type

$$H_I = i\hbar \int d^3k \{ \theta(\mathbf{k}) \otimes a^{\dagger}(\mathbf{k}) - \theta^{\dagger}(\mathbf{k}) \otimes a(\mathbf{k}) \}, \quad (33)$$

where $\{\theta(\mathbf{k}) : \mathbf{k} \in \mathbb{R}^3\}$ is a family of operators on \mathcal{H}_S . The operators $\theta(\mathbf{k})$ are called the response terms: they contain local information about the interaction. In the dipole approximation of quantum field theory one makes the replacement

$$\theta(\mathbf{k}) \hookrightarrow \theta^{\text{dipole}}(\mathbf{k}) = g(\mathbf{k})\theta(0), \quad (34)$$

where $g(\mathbf{k})$ is some suitable test function. The physical argument is, cf. [2], that the response does not vary appreciably for values of the wavelength of the reservoir particles which are large relative to the physical dimensions of the system, though this can hardly be true for large momenta. As a result one obtains the approximate Hamiltonian

$$H_I \hookrightarrow H_I^{\text{dipole}} = i\hbar \{ \theta(0) \otimes A^{\dagger}(g) - \theta^{\dagger}(0) \otimes A(g) \}. \quad (35)$$

A further approximation often made by physicists is to replace $\theta(0)$ by an operator D having a harmonic free-evolution with some frequency $\omega \in \mathbb{R}$. This approximation is just the rotating wave approximation.

In order to avoid these approximations we argue as follows: Let B be a complete basis of eigenstates of H_S , then

$$H_I = \sum_{\phi, \phi' \in B} \langle \phi | H_I | \phi' \rangle | \phi \rangle \langle \phi' |, \quad (36)$$

but however we may write

$$\langle \phi | H_I | \phi' \rangle = i\hbar \int dk \{ \langle \phi | \theta(\mathbf{k}) | \phi' \rangle a^{\dagger}(\mathbf{k}) - \langle \phi | \theta^{\dagger}(\mathbf{k}) | \phi' \rangle a(\mathbf{k}) \}$$

$$\equiv i\hbar[A^\dagger(g_{\phi\phi'}) - A(g_{\phi'\phi})], \quad (37)$$

where we have introduced the test functions

$$g_{\phi\phi'}(\mathbf{k}) = \langle \phi | \theta(\mathbf{k}) | \phi' \rangle : \quad (38)$$

note that the order of ϕ and ϕ' is reversed in the second term in due to the conjugate linear nature of the creation field. This now means that the interaction can be expressed as

$$H_I = i\hbar \sum_{\phi, \phi' \in B} \{T_{\phi\phi'} \otimes A^\dagger(g_{\phi\phi'}) - T_{\phi\phi'}^\dagger \otimes A(g_{\phi\phi'})\}, \quad (39)$$

where we have introduced the transition operators $T_{\phi\phi'} = |\phi\rangle\langle\phi'|$.

We note that the transition operators $T_{\phi\phi'}$ are harmonic under the free evolution, in fact we have

$$\frac{1}{i\hbar}[T_{\phi\phi'}, H_S] = -i\omega_{\phi\phi'} T_{\phi\phi'}, \quad (40)$$

where

$$\omega_{\phi\phi'} = \frac{E_{\phi'} - E_{\phi}}{\hbar}. \quad (41)$$

So $F = \{\omega_{\phi\phi'} : \phi, \phi' \in B\}$ is now the set of Bohr frequencies.

The expression (38) is now equivalent to the interaction (30) which we propose to study. Here we need only relabel the $T_{\phi\phi'}$ as D_j^ω where $\omega = \omega_{\phi\phi'}$ and the j again labels degeneracy. The functions $g_{\phi\phi'}$ are relabeled accordingly.

2 THE QUANTUM STOCHASTIC LIMIT

2.1 Quantum Brownian Motions

In this section we first of all discuss the concept of quantum brownian motion. As this is not yet widely known amongst physicists we give an exposition below:

Definition. A quantum brownian motion (QBM) is a triple $(H, \Phi, (B_t)_t)$, where H is a separable Hilbert space, $\Phi \in H$ with $\|\Phi\| = 1$, $(B_t)_t$ is a family of operators on H such that

(i) $q_t = \text{Re} B_t$ and $p_t = \text{Im} B_t$ are classical brownian motions for the state Φ .

(ii) $[p_s, q_t] = \frac{\kappa}{2i} \min(s, t)$, where $\kappa \in \mathbb{R}$.

The basic example is the following: Let $\mathcal{H}_\mathbb{C} = \Gamma_B(L^2(\mathbb{R}))$ and $\Phi_\mathbb{C}$ be the vacuum state. Then define B_t to be

$$B_t = A_\mathbb{C}(\chi_{[0,t]}) \quad (42)$$

where $A_\mathbb{C}$ is the annihilation operator on $\mathcal{H}_\mathbb{C}$. From the (CCR) we have

$$[B_t, B_s^\dagger] = \langle \chi_{[0,t]}, \chi_{[0,s]} \rangle = \min(t, s); [B_t, B_s] = 0 = [B_t^\dagger, B_s^\dagger]. \quad (43)$$

So setting $q_t = \frac{1}{2}(B_t + B_t^\dagger)$ and $p_t = \frac{1}{2i}(B_t - B_t^\dagger)$ we have from the (CCR) that $[p_t, q_s] = -[q_t, p_s]$ and

$$[p_t, q_s] = \frac{1}{2i} \min(t, s). \quad (44)$$

Now if we set

$$dB_t = B_{t+dt} - B_t = A_\mathbb{C}(\chi_{[t, t+dt]}), \quad (45)$$

we have that

$$\begin{aligned} \langle \Phi_\mathbb{C}, dB_t^\sharp \Phi_\mathbb{C} \rangle &= 0; \langle \Phi_\mathbb{C}, (dB_t^\sharp)^2 \Phi_\mathbb{C} \rangle = 0; \langle \Phi_\mathbb{C}, dB_t^\dagger dB_t \Phi_\mathbb{C} \rangle = 0; \\ \langle \Phi_\mathbb{C}, dB_t dB_t^\dagger \Phi_\mathbb{C} \rangle &= dt. \end{aligned} \quad (46)$$

Now dB_t and dB_t^\dagger are gaussian in the vacuum state, because the creation and annihilation fields are, therefore so too are dq_t and dp_t . Furthermore,

$$\langle \Phi_\mathbb{C}, (dq_t)^2 \Phi_\mathbb{C} \rangle = \frac{1}{4} \langle \Phi_\mathbb{C}, (dB_t + dB_t^\dagger)^2 \Phi_\mathbb{C} \rangle = \frac{1}{4} dt \quad (47)$$

and similarly $\langle \Phi_{\mathbb{C}}, (dp_t)^2 \Phi_{\mathbb{C}} \rangle = \frac{1}{4} dt$. Finally noting that at unequal times s and t

$$\langle \Phi_{\mathbb{C}}, dq_t dq_s \Phi_{\mathbb{C}} \rangle = 0 = \langle \Phi_{\mathbb{C}}, dp_t dp_s \Phi_{\mathbb{C}} \rangle, \quad (48)$$

whenever $t(t+dt) \leq s(s+ds)$ or $s(s+ds) \leq t(t+dt)$, we conclude that $(q_t)_t$ and $(p_t)_t$ are each separate Brownian motions for expectations taken in the state $\Phi_{\mathbb{C}}$. So $\{\mathcal{H}_{\mathbb{C}}, \Phi_{\mathbb{C}}, (B_t)_t\}$ is a QBM. We can introduce formal creation and annihilation densities $b^\sharp(t)$ satisfying

$$[b(t), b(s)] = 0 = [b^+(t), b^+(s)]; [b(t), b^+(s)] = \delta(t-s), \quad (49)$$

such that

$$A_{\mathbb{C}}(g) = \int_{\mathbb{R}} ds \overline{g(s)} b(s); A_{\mathbb{C}}^\dagger(g) = \int_{\mathbb{R}} ds g(s) b^+(s); \quad (50)$$

from this we see

$$B_t^\sharp = \int_0^t ds b^\sharp(s). \quad (51)$$

We may write $b_t^\sharp = \frac{dB_t}{dt}$ and consider these densities as “quantum white noises”.

More generally let K be a separable Hilbert space and let $L^2(\mathbb{R}, K)$ denote the set of square-integrable K -valued functions over \mathbb{R} . Now $h \in L^2(\mathbb{R}, K)$ is a function $h(t) \in K$ with $\int_{\mathbb{R}} dt \|h(t)\|_K^2 < \infty$. The inner product on $L^2(\mathbb{R}, K)$ is given by

$$\langle h, h' \rangle = \int_{\mathbb{R}} \langle h(t), h'(t) \rangle_K dt. \quad (52)$$

If $\{e_n\}_n$ is a complete orthonormal basis for K then we can write $h(t) = \sum_n h_n(t) e_n$, where $h_n(t) = \langle e_n, h(t) \rangle_K$; this gives a natural isomorphism

$$L^2(\mathbb{R}, K) \cong K \otimes L^2(\mathbb{R}). \quad (53)$$

Now take $\mathcal{H}_K = \Gamma_B(L^2(\mathbb{R}, K))$ and let Φ_K denote vacuum vector of \mathcal{H}_K , then a QBM is given by $(\mathcal{H}_K, \Phi_K, (B_t(g))_t)$, for non-zero $g \in K$, where

$$B_t(g) = A_K(g \otimes \chi_{[0,t]}) \quad (54)$$

where A_K is the annihilation operator on \mathcal{H}_K . The commutation relations are

$$[B_t(g), B_s^\dagger(f)] = \langle g \otimes \chi_{[0,t]}, f \otimes \chi_{[0,s]} \rangle = \langle g, f \rangle_K \min(t, s), \quad (55)$$

with remaining commutators vanishing. So $\{\mathcal{H}_K = \Gamma_B(L^2(\mathbb{R}, K)), \Phi_K, (B_t(g))_t\}$ is a QBM. Taking $K = \mathbb{C}$ and $|g|^2 = \kappa$ leads back to the original example.

However, there is a more general possibility than that above; Let $Q \geq 1_K$ and set

$$C = Q \otimes 1_{L^2(\mathbb{R})}. \quad (56)$$

Then let φ_C be the state on the Weyl algebra $W(\mathcal{H}_K)$ with covariance C . We can construct $\{G_B(\mathcal{H}_K, C), \pi_{\mathcal{H}_K}^C, \Phi_{\mathcal{H}_K}^C\}$ the GNS triple over $\{W(\mathcal{H}_K), C\}$ and on it define the operator

$$B_Q(g, t) = \pi_{\mathcal{H}_K}^C B_t(g). \quad (57)$$

Then $\{G_B(\mathcal{H}_K, C), \Phi_{\mathcal{H}_K}^C, (B_Q(g, t))_t\}$ is a QBM referred to as quantum Brownian motion over $L^2(\mathbb{R}, K)$ with covariance C , or more loosely with covariance Q . We have that

$$\langle \Phi_{\mathcal{H}_K}^C, B_Q(g, t) B_Q^\dagger(f, s) \Phi_{\mathcal{H}_K}^C \rangle = \varphi_C(B_t(g) B_s^\dagger(f)) = \min(t, s) \langle g, \frac{Q+1}{2} f \rangle_K, \quad (58)$$

and similarly

$$\langle \Phi_{\mathcal{H}_K}^C, B_Q^\dagger(f, s) B_Q(g, t) \Phi_{\mathcal{H}_K}^C \rangle = \min(t, s) \langle g, \frac{Q-1}{2} f \rangle_K. \quad (59)$$

2.2 Quantum Stochastic Calculus

As is well known, a stochastic calculus can be built up around classical brownian motion and that the resulting theory has widespread applications to studying noisy systems in physics and engineering. It is also possible to build up a quantum stochastic calculus based on the QBMs we have just considered. This was originally done by Hudson and Parthasarathy [13], [14]. The basic integrators are dt and, depending on the context, dB_t^\sharp or $dB_t^\sharp(g)$ or $dB_Q^\sharp(t, g)$.

In the simplest case, for instance, we have, for a partition $-\infty = t_1 < t_2 < \dots < t_n < t_{n+1} = \infty$,

$$L^2(\mathbb{R}) = \bigoplus_{m=1}^n L^2([t_m, t_{m+1}])$$

and consequently

$$\Gamma_B(L^2(\mathbb{R})) = \bigotimes_{m=1}^n \Gamma_B(L^2([t_m, t_{m+1}])). \quad (60)$$

This gives the required time filtration in the quantum situation. We say that a family of operators $(X_t)_t$ on $\Gamma_B(L^2(\mathbb{R}))$ is adapted if, for all t ,

$$X_t \equiv \tilde{X}_t \otimes 1 \quad (61)$$

on $\Gamma_B(L^2((-\infty, t))) \otimes \Gamma_B(L^2([t, \infty)))$. The quantum Ito table reads as

$$\begin{aligned} dB_Q(g, t) dB_Q^\dagger(f, t) &\equiv \langle g, \frac{Q+1}{2} f \rangle_K dt, \\ dB_Q^\dagger(f, t) dB_Q(g, t) &\equiv \langle g, \frac{Q-1}{2} f \rangle_K dt, \\ (dt)^2, dt dB_Q^\sharp(g, t), (dB_Q^\sharp(g, t))^2 &\equiv 0, \end{aligned} \quad (62)$$

Let $(X_t)_t$ be an adapted process and of the form

$$X_t = \int_0^t (x_s ds + x_s^+ dB_Q^\dagger(g, s) + x_s^- dB_Q(g, s)) \quad (63)$$

and $(Y_t)_t$ a similar process, then we have the quantum Ito formula

$$d(X_t \cdot Y_t) \equiv dX_t \cdot Y_t + X_t \cdot dY_t + dX_t \cdot dY_t \quad (64)$$

with

$$dX_t = x_t dt + x_t^+ dB_Q^\dagger(g, t) + x_t^- dB_Q(g, t). \quad (65)$$

2.3 The Weak Coupling Limit of Quantum Field Theory

The results of Accardi, Frigerio and Lu concerning the weak coupling limit for an interaction (31) which has undergone both a dipole and rotating wave approximation can be summarised as follows;

Recall that

$$\lim_{\lambda \rightarrow 0} \langle \Psi_R, B_{t/\lambda^2}^{(\omega, \lambda)}(g) B_{s/\lambda^2}^{(\omega, \lambda)\dagger}(f) \Psi_R \rangle = \min(t, s) \int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, f \rangle;$$

now define a sesquilinear form $(\cdot | \cdot)^\omega$ on \mathcal{H}_R^1 the one particle reservoir space by

$$(g | f)^\omega = \int_{-\infty}^{\infty} d\tau \langle S_\tau^\omega g, f \rangle. \quad (66)$$

We consider the space of suitable test-functions $T^\omega \subset \mathcal{H}_R^1$, determined by the condition

$$\int_{-\infty}^{\infty} dt |\langle g, S_t^\omega f \rangle| < \infty, \quad (67)$$

whenever $f, g \in T^\omega$. Note that technically T^ω does not depend on ω however we keep it in as a label. Then we construct K_ω the completion of T^ω with respect to $(\cdot | \cdot)^\omega$. That is K^ω is the completion of T^ω factored out by its $(\cdot | \cdot)^\omega$ -norm null space. K^ω is a separable Hilbert space with inner product $(\cdot | \cdot)^\omega$

Theorem 1. In the limit $\lambda \rightarrow 0$ the stochastic process on the reservoir space

$$\{\mathcal{H}_R, \Psi_R, (B_{t/\lambda^2}^{(\omega, \lambda)}(f))_t\},$$

for $f \in K^\omega$, converges weakly in the sense of matrix elements to QBM on $L^2(R, K_\omega)$. We denote this QBM by $\{H^\omega = \Gamma_B(L^2(R, K^\omega)), \Phi^\omega = \Phi_{K^\omega}, (B_t^\omega(f))_t\}$. In the next theorem we show that $U_{t/\lambda^2}^{(\lambda)}$ converges to a stochastic process U_t on $H_S \otimes H^\omega$ in a sense to be made explicit now.

Theorem 2. Let $f^{(j)}, h^{(j')} \in K^\omega$; $T^{(j)}, S^{(j')} \geq 0$, for $j = 1, \dots, n$; $j' = 1, \dots, m$ and let $\phi, \phi' \in H_S$ then the limit as $\lambda \rightarrow 0$ of the matrix element

$$\langle \phi \otimes B_{T^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(f^{(1)}) \dots B_{T^{(n)}/\lambda^2}^{(\omega, \lambda)\dagger}(f^{(n)}) \Psi_R | U_{t/\lambda^2}^{(\lambda)} | \phi' \otimes B_{S^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(1)}) \dots B_{S^{(m)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(m)}) \Psi_R \rangle \quad (68)$$

exists and equals

$$\langle \phi \otimes B_{T^{(1)}}^{\omega\dagger}(f^{(1)}) \dots B_{T^{(n)}}^{\omega\dagger}(f^{(n)}) \Phi^\omega | U_t | \phi' \otimes B_{S^{(1)}}^{\omega\dagger}(h^{(1)}) \dots B_{S^{(m)}}^{\omega\dagger}(h^{(m)}) \Phi^\omega \rangle, \quad (69)$$

where U_t is a process on $H_S \otimes H^\omega$ which is the solution to the quantum stochastic differential equation

$$dU_t = \{D \otimes dB_t^\omega(g) - D^\dagger \otimes dB_t^\omega(g) - (g|g)^{\omega-} D^\dagger D \otimes dt\} U_t, \quad (70)$$

with

$$(g|f)^{\omega-} = \int_{-\infty}^0 d\tau \langle g, S_\tau^\omega f \rangle. \quad (71)$$

Note that $d(U_t U_t^\dagger) \equiv 0 \equiv d(U_t^\dagger U_t)$ by the quantum Ito formula and the Ito table. So U_t is *unitary* on $\mathcal{H}_S \otimes \mathcal{H}^\omega$, however it describes an *irreversible* evolution when restricted to \mathcal{H}_S . The unitarity condition corresponds to a fluctuation-dissipation law.

Theorem 3 Let $X \in B(H_S)$, then in the notation of theorem 2 the limit

$$\begin{aligned} & \langle \phi \otimes B_{T^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(f^{(1)}) \dots B_{T^{(n)}/\lambda^2}^{(\omega, \lambda)\dagger}(f^{(n)}) \Psi_R | \\ & U_{t/\lambda^2}^{(\lambda)\dagger} (X \otimes 1_R) U_{t/\lambda^2}^{(\lambda)} | \phi' \otimes B_{S^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(1)}) \dots B_{S^{(m)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(m)}) \Psi_R \rangle \end{aligned} \quad (72)$$

exists and equals

$$\langle \phi \otimes B_{T^{(1)}}^{\omega\dagger}(f^{(1)}) \dots B_{T^{(n)}}^{\omega\dagger}(f^{(n)}) \Phi^\omega | U_t^\dagger (X \otimes 1) U_t | \phi' \otimes B_{S^{(1)}}^{\omega\dagger}(h^{(1)}) \dots B_{S^{(m)}}^{\omega\dagger}(h^{(m)}) \Phi^\omega \rangle. \quad (73)$$

Note that in these theorems we encounter vectors of the type $B_{T/\lambda^2}^{(\omega, \lambda)\dagger}(f) \Psi_R$ which are exponential vectors with test functions $\lambda \int_0^{T/\lambda^2} d\tau S_\tau^\omega f$: they are called *collective coherent vectors* in the terminology of Accardi and Lu and they are designed to extract the long time cumulative behaviour of the reservoir fields.

2.4 Non-Zero Temperature Reservoir

Next, for the non-vacuum case, we consider a density matrix ρ_Q on \mathcal{H}_R which is invariant under the free evolution and gaussian with covariance $Q \geq 1_{\mathcal{H}_R^1}$. That is

$$\text{Tr}\{\rho_Q W(g)\} = e^{-\frac{1}{2}\langle g, Qg \rangle}, \quad \forall g \in \mathcal{H}_R^1. \quad (74)$$

The invariance condition is equivalent to

$$[S_t, Q] = 0, \text{ on } \text{Dom}(Q). \quad (75)$$

In particular, the choice of a heat bath at inverse temperature β and fugacity z is given by

$$Q = \frac{1 + ze^{-\beta H_R^1}}{1 - ze^{-\beta H_R^1}}, \quad (76)$$

that is

$$(Qf)(k) = \coth \frac{\beta}{2} (\hbar\omega(k) - \mu) f(k), \quad (77)$$

where $\mu = \frac{1}{\beta} \ln z$ is the chemical potential.

Now

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \text{Tr}\{\rho_Q B_{t/\lambda^2}^{(\omega, \lambda)}(g) B_{s/\lambda^2}^{(\omega, \lambda)\dagger}(f)\} &= \min(t, s) \int_{-\infty}^{\infty} d\tau \text{Tr}\{\rho_Q A(S_{\tau}^{\omega} g) A^{\dagger}(f)\} \\ &= \min(t, s) \int_{-\infty}^{\infty} d\tau \langle S_{\tau}^{\omega} g, (\frac{Q+1}{2}) f \rangle, \end{aligned} \quad (78)$$

and similarly

$$\lim_{\lambda \rightarrow 0} \text{Tr}\{\rho_Q B_{s/\lambda^2}^{(\omega, \lambda)\dagger}(f) B_{t/\lambda^2}^{(\omega, \lambda)}(g)\} = \min(t, s) \int_{-\infty}^{\infty} d\tau \langle S_{\tau}^{\omega} g, (\frac{Q-1}{2}) f \rangle. \quad (79)$$

Let T_Q^{ω} be the subset of $\text{Dom}(Q)$ such that

$$\int_{-\infty}^{\infty} |\langle f, S_t^{\omega} h \rangle| dt < \infty \text{ and } \int_{-\infty}^{\infty} |\langle f, S_t^{\omega} Q h \rangle| dt < \infty, \quad (80)$$

whenever $f, h \in T_Q^{\omega}$. Let K_Q^{ω} be the Hilbert space completion of T_Q^{ω} with respect to the sesquilinear form $(\cdot|\cdot)_Q^{\omega}$ given by

$$(f|h)_Q^{\omega} = \int_{-\infty}^{\infty} \langle f, Q S_t^{\omega} h \rangle dt. \quad (81)$$

Note that in most cases T_Q^{ω} is dense in \mathcal{H}_R^1 and that K_Q^{ω} is a Hilbert space equipped with inner product $(\cdot|\cdot)_Q^{\omega}$.

Theorem 1a: *The process $(B_{t/\lambda^2}^{(\lambda)}(f))_t$ in the mixed state ρ_Q converges weakly in the sense of matrix elements to QBM over $L^2(R, K_Q^{\omega})$ with covariance Q . This is denoted as $\{H_Q^{\omega} = G_B(L^2(R, K_Q^{\omega}), Q \otimes 1), \Phi_Q^{\omega}, (B_Q^{\omega}(f, t))_t\}$.*

Theorem 2a: *For $\phi, \phi' \in H_S$, $f^{(j)}, h^{(j')} \in K_Q^{\omega}$ and $T^{(j)}, S^{(j')} \geq 0$, for $j = 1, \dots, n; j' = 1, \dots, m$, the limit as $\lambda \rightarrow 0$ of*

$$\text{Tr}\{|\phi'\rangle\langle\phi| \otimes B_{S^{(m)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(m)}) \dots B_{S^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(1)}) \rho_Q B_{T^{(1)}/\lambda^2}^{(\omega, \lambda)}(f^{(1)}) \dots B_{T^{(n)}/\lambda^2}^{(\omega, \lambda)}(f^{(n)}) U_{t/\lambda^2}^{(\lambda)}\} \quad (82)$$

exists and equals

$$\langle \phi \otimes B_Q^{\omega\dagger}(f^{(1)}, T^{(1)}) \dots B_Q^{\omega\dagger}(f^{(n)}, T^{(n)}) \Phi_Q^{\omega} | U_t | \phi' \otimes B_Q^{\omega\dagger}(h^{(1)}, S^{(1)}) \dots B_Q^{\omega\dagger}(h^{(m)}, S^{(m)}) \Phi_Q^{\omega} \rangle \quad (83)$$

where U_t is a unitary operator on $H_S \otimes G_B(L^2(R, K_Q^{\omega}), C)$, with $C = Q \otimes 1$, satisfying the QSDE

$$dU_t = \{D \otimes dB_Q^{\omega\dagger}(g, t) - D^{\dagger} \otimes dB_Q^{\omega}(g, t) - (g|g)_{Q+}^{\omega-} D^{\dagger} D \otimes dt - \overline{(g|g)}_{Q-}^{\omega-} D D^{\dagger} \otimes dt\} U_t, \quad (84)$$

where

$$(g|f)_{Q\pm}^{\omega-} = \int_{-\infty}^0 d\tau \langle g, S_{\tau}^{\omega} \frac{Q \pm 1}{2} f \rangle. \quad (85)$$

Theorem 3a: *For $X \in B(H_S)$, then in the notation of theorem 2*

$$\lim_{\lambda \rightarrow 0} \text{Tr}\{|\phi'\rangle\langle\phi| \otimes B_{S^{(m)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(m)}) \dots B_{S^{(1)}/\lambda^2}^{(\omega, \lambda)\dagger}(h^{(1)}) \rho_Q B_{T^{(1)}/\lambda^2}^{(\omega, \lambda)}(f^{(1)}) \dots B_{T^{(n)}/\lambda^2}^{(\omega, \lambda)}(f^{(n)}) u_{t/\lambda^2}^{(\lambda)}(X \otimes 1)\} \quad (86)$$

exists and equals

$$\langle \phi \otimes B_Q^{\omega\dagger}(f^{(1)}, T^{(1)}) \dots B_Q^{\omega\dagger}(f^{(n)}, T^{(n)}) \Phi_Q^{\omega} | U_t^{\dagger}(X \otimes 1) U_t | \phi' \otimes B_Q^{\omega\dagger}(h^{(1)}, S^{(1)}) \dots B_Q^{\omega\dagger}(h^{(m)}, S^{(m)}) \Phi_Q^{\omega} \rangle. \quad (87)$$

2.5 The Quantum Stochastic Limit for the Full Interaction

Now suppose that the interaction is of the form (31). The problem of dropping the rotating wave approximation was first tackled by Accardi and Lu in [10] for an interaction similar to (31) except that all the test functions were taken to be the same. The result is that for each Bohr frequency ω we obtain a separate independent QBM.

First of all note that $\Gamma_B(\bigoplus_{\omega \in F} L^2(\mathbb{R}, K_Q^\omega)) = \bigotimes_{\omega \in F} \Gamma_B(L^2(\mathbb{R}, K_Q^\omega))$ then consider the Weyl algebra $W(\bigoplus_{\omega \in F} L^2(\mathbb{R}, K_Q^\omega)) = \bigotimes_{\omega \in F} W(L^2(\mathbb{R}, K_Q^\omega))$, with quasi-free state $\varphi_{\tilde{C}}$ with covariance $\tilde{C} = \bigotimes_{\omega \in F} C$, where $C = Q \otimes 1$ on $\mathcal{H}_{K_Q^\omega} \cong K_Q^\omega \otimes L^2(\mathbb{R})$. The GNS triple over $\{W(\bigoplus_{\omega \in F} L^2(\mathbb{R}, K_Q^\omega)), \varphi_{\tilde{C}}\}$ is $\{\mathcal{H}_Q^\omega = \Gamma_B(\bigoplus_{\omega \in F} L^2(\mathbb{R}, K_Q^\omega)), \tilde{C}\}, \pi_Q^F, \Phi_Q^\omega\}$. Now observe that

$$\mathcal{H}_Q^F = \bigotimes_{\omega \in F} \mathcal{H}_Q^\omega, \pi_Q^F = \bigotimes_{\omega \in F} \pi_Q^\omega; \Phi_Q^F = \bigotimes_{\omega \in F} \Phi_Q^\omega. \quad (88)$$

For each $f \in K_Q^\omega$ we have

$$B_Q^\omega(f, t) = \pi_Q^\omega A_Q^\omega(f \otimes \chi_{[0, t]}), \quad (89)$$

where A_Q^ω is annihilation operator on $L^2(\mathbb{R}, K_Q^\omega)$, so for $t_\omega > 0, f_\omega \in K_Q^\omega$, for each $\omega \in F$, we have

$$B_Q^F(\bigotimes_{\omega \in F} f_\omega, (t_\omega)_{\omega \in F}) = \bigotimes_{\omega \in F} B_Q^\omega(f_\omega, t_\omega). \quad (90)$$

Theorem 1b: For each $\omega \in F$ and $f \in K_Q^\omega$ the limit $\lambda \rightarrow 0$, $B_{t/\lambda^2}^{(\omega, \lambda)}(f)$ taken in the state ρ_Q converges in the sense of matrix elements to QBM over $L^2(\mathbb{R}, K_Q^\omega)$ with covariance Q and each of these limiting processes are independent for different values of ω .

Theorem 2b: Let $f_\omega^{(j)}, h_\omega^{(j')} \in K_Q^\omega, T_\omega^{(j)}, S_\omega^{(j')} > 0$ for each $\omega \in F, j = 1, \dots, n_\omega; j' = 1, \dots, m_\omega$; and $t \geq 0$; $\phi, \phi' \in H_S$ then the limit as $\lambda \rightarrow 0$ of the matrix element

$$\text{Tr}\{|\phi'\rangle\langle\phi| \otimes [\bigotimes_{\omega \in F} B_{S_\omega^{(m_\omega)}/\lambda^2}^{(\omega, \lambda)}(h_\omega^{(m_\omega)}) \dots B_{S_\omega^{(1)}/\lambda^2}^{(\omega, \lambda)}(h_\omega^{(1)})]^\dagger \times \rho_Q[\bigotimes_{\omega \in F} B_{T_\omega^{(1)}/\lambda^2}^{(\omega, \lambda)}(f_\omega^{(1)}) \dots B_{T_\omega^{(n_\omega)}/\lambda^2}^{(\omega, \lambda)}(f_\omega^{(n_\omega)})] U_{t/\lambda^2}^{(\lambda)}\} \quad (91)$$

exists and equals

$$\langle\phi \otimes [\bigotimes_{\omega \in F} B_Q^\omega(f_\omega^{(1)}, T_\omega^{(1)}) \dots B_Q^\omega(f_\omega^{(n_\omega)}, T_\omega^{(n_\omega)})]^\dagger \Phi_Q^F| \quad (92)$$

$$U_t |\phi' \otimes [\bigotimes_{\omega \in F} B_Q^\omega(h_\omega^{(1)}, S_\omega^{(1)}) \dots B_Q^\omega(h_\omega^{(m_\omega)}, S_\omega^{(m_\omega)})]^\dagger \Phi_Q^F\rangle, \quad (93)$$

where U_t is unitary on $H_S \otimes H_Q^F$ and satisfies quantum stochastic differential equation

$$\begin{aligned} dU_t = & \left\{ \sum_{\omega \in F} \sum_{j=1}^{N(\omega)} [D_j^\omega \otimes dB_\omega^\dagger(g_j^\omega, t) - D_j^{\omega\dagger} \otimes dB_\omega(g_j^\omega, t)] \right. \\ & \left. - \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} D_j^{\omega\dagger} D_k^\omega (g_j^\omega | g_k^\omega)_{Q+}^{\omega-} dt - \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} D_j^\omega D_k^{\omega\dagger} \overline{(g_j^\omega | g_k^\omega)_{Q-}^{\omega-}} dt \right\} U_t, \end{aligned} \quad (94)$$

with $U_0 = 1$. The coefficients are given by

$$(f|h)_{Q\pm}^{\omega-} = \int_{-\infty}^0 dt \langle f, S_t^\omega(\frac{Q \pm 1}{2}) h \rangle, \quad (95)$$

for $f, h \in K_\omega$. The Ito table is given by

$$\begin{aligned} dB_Q^\omega(f, t) dB_Q^{\omega'\dagger}(g, t) &\equiv \delta_{\omega, \omega'} (f|g)_{Q+}^\omega dt \\ dB_Q^{\omega\dagger}(g, t) dB_Q^{\omega'}(f, t) &\equiv \delta_{\omega, \omega'} (f|g)_{Q-}^\omega dt. \end{aligned} \quad (96)$$

The coefficients in (2.5:8) are

$$(f|g)_{Q+}^\omega = \int_{-\infty}^\infty dt \langle f, S_t^\omega(\frac{Q+1}{2}) g \rangle = (f|g)_{Q+}^{\omega-} + \overline{(g|f)_{Q+}^{\omega-}};$$

$$(f|g)_{Q-}^{\omega} = \int_{-\infty}^{\infty} dt \langle f, S_t^{\omega}(\frac{Q-1}{2})g \rangle = \overline{(g|f)_{Q-}^{\omega-}} + (f|g)_{Q-}^{\omega-}. \quad (97)$$

The proof of theorem 2a is as follows; first of all we know that for different ω we can always consider independent Q-quantum Brownian motions. This is done in [10]. The next step is to consider the effect of the degeneracy which may arise for each $\omega \in F$. In this case we must, therefore, generalize the results of [8],[10] accordingly. This is done in appendices B and C.

Theorem 3a. *In the notations of theorem 2a, for any $X \in B(H_S)$, the limit as $\lambda \rightarrow 0$*

$$\text{Tr}\{|\phi'\rangle\langle\phi| \otimes [\bigotimes_{\omega \in F} B_{S_{\omega}^{(1)}/\lambda^2}^{(\omega,\lambda)}(h_{\omega}^{(1)}) \dots B_{S_{\omega}^{(m_{\omega})}/\lambda^2}^{(\omega,\lambda)}(h_{\omega}^{(m_{\omega})})]^{\dagger} \rho_Q[\bigotimes_{\omega \in F} B_{T_{\omega}^{(1)}/\lambda^2}^{(\omega,\lambda)}(f_{\omega}^{(1)}) \dots B_{T_{\omega}^{(n_{\omega})}/\lambda^2}^{(\omega,\lambda)}(f_{\omega}^{(n_{\omega})})] u_{t/\lambda^2}^{(\lambda)}(X \otimes 1)\} \quad (98)$$

exists and equals

$$\langle \phi \otimes [\bigotimes_{\omega \in F} B_Q^{\omega}(f_{\omega}^{(1)}, T_{\omega}^{(1)}) \dots B_Q^{\omega}(f_{\omega}^{(n_{\omega})}, T_{\omega}^{(n_{\omega})})]^{\dagger} \Phi_Q^F | U_t^{\dagger}(X \otimes 1) U_t | \phi' \otimes [\bigotimes_{\omega \in F} B_Q^{\omega}(h_{\omega}^{(1)}, S_{\omega}^{(1)}) \dots B_Q^{\omega}(h_{\omega}^{(m_{\omega})}, S_{\omega}^{(m_{\omega})})]^{\dagger} \Phi_Q^F \rangle, \quad (99)$$

where U_t is the solution to the quantum stochastic differential equation of theorem.

2.6 The Langevin and Master Equations

In each of the cases the right hand side of the expression for dU_t contains a term of the form $-Y \otimes dtU_t$. For instance, we have

$$Y = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ D_j^{\omega\dagger} D_k^{\omega} (g_j^{\omega} | g_k^{\omega})_{Q+}^{\omega-} + D_j^{\omega} D_k^{\omega\dagger} \overline{(g_j^{\omega} | g_k^{\omega})_{Q-}^{\omega-}} \}, \quad (100)$$

The Langevin equation then reads as follows

$$\begin{aligned} d[U_t^{\dagger}(X \otimes 1)U_t] &\equiv [dU_t]^{\dagger}(X \otimes 1)U_t + U_t^{\dagger}(X \otimes 1)dU_t + [dU_t]^{\dagger}(X \otimes 1)dU_t \\ &\equiv U_t^{\dagger}[L_0(X) \otimes dt + \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} L_{j+}^{\omega}(X) \otimes dB_Q^{\omega\dagger}(g_j^{\omega}, t) \\ &\quad + \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} L_{j-}^{\omega}(X) \otimes dB_Q^{\omega}(g_j^{\omega}, t)]U_t, \end{aligned} \quad (101)$$

where

$$L_0(X) = -XY - Y^{\dagger}X + \Theta(X), \quad (102)$$

with

$$\begin{aligned} \Theta(X) &= \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ D_j^{\omega\dagger} X D_k^{\omega} [(g_j^{\omega} | g_k^{\omega})_{Q+}^{\omega-} + \overline{(g_k^{\omega} | g_j^{\omega})_{Q+}^{\omega-}}] + D_j^{\omega} X D_k^{\omega\dagger} [\overline{(g_j^{\omega} | g_k^{\omega})_{Q-}^{\omega-}} + (g_k^{\omega} | g_j^{\omega})_{Q-}^{\omega-}] \} \\ &= \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ D_j^{\omega\dagger} X D_k^{\omega} (g_j^{\omega} | g_k^{\omega})_{Q+}^{\omega} + D_k^{\omega} X D_j^{\omega\dagger} (g_j^{\omega} | g_k^{\omega})_{Q-}^{\omega} \}, \end{aligned} \quad (103)$$

and

$$L_{j+}^{\omega}(X) = X D_j^{\omega} - D_j^{\omega} X, \quad L_{j-}^{\omega}(X) = D_j^{\omega\dagger} X - X D_j^{\omega\dagger}. \quad (104)$$

Note that unitarity follows from

$$L_0(1_S) = -(Y + Y^{\dagger}) + \Theta(1_S) = 0; \quad L_{j\pm}^{\omega}(1_S) = 0. \quad (105)$$

It is instructive to set $Y = \frac{1}{2}\Gamma + \frac{i}{\hbar}H'_S$ where both Γ and H'_S are self-adjoint; we then have that

$$L_0(X) = -\frac{1}{2}(X\Gamma + \Gamma X) + \Theta(X) + \frac{1}{i\hbar}[X, H'_S]. \quad (106)$$

The unitarity condition is then $2\text{Re } Y = \Gamma = \Theta(1_S)$; this is the fluctuation-dissipation relation. The presence of the imaginary term H'_S does not effect the unitarity. For ρ_S a density matrix on \mathcal{H}_S we define the expectation $\langle \cdot \rangle_t$ by

$$\langle X \rangle_t = \text{Tr}\{\rho_S \otimes |\Phi_Q^F\rangle\langle\Phi_Q^F| U_t^\dagger (X \otimes 1) U_t\} = \text{Tr}\{s_t X\}, \quad (107)$$

where s_t denotes the effective density matrix on (S) and the second trace is a partial trace over the system space (the trace over the reservoir space assumed to be taken already). Now

$$\frac{d}{dt}\langle X \rangle_t = \text{Tr}\{\rho_S \otimes |\Phi_Q^F\rangle\langle\Phi_Q^F| \frac{d}{dt} U_t^\dagger (X \otimes 1) U_t\}, \quad (108)$$

so in terms of the effective density matrix s_t

$$\frac{d}{dt}\text{Tr}\{s_t X\} = \text{Tr}\{s_t L_0(X)\} = \text{Tr}\{L_0^*(s_t) X\}, \quad (109)$$

where L_0^* denotes the adjoint operation to L_0 on the dual of $\mathcal{B}(\mathcal{H}_S)$. This gives the master equation

$$\begin{aligned} \frac{ds_t}{dt} &= L_0^*(s_t) = - \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ [D_j^\omega D_k^{\omega\dagger} s_t - D_k^{\omega\dagger} s_t D_j^\omega] \overline{(g_j^\omega | g_k^\omega)_{Q-}^{\omega-}} + [D_j^{\omega\dagger} D_k^\omega s_t - D_k^\omega s_t D_j^{\omega\dagger}] (g_j^\omega | g_k^\omega)_{Q+}^{\omega-} \\ &\quad - [D_j^{\omega\dagger} s_t D_k^\omega - s_t D_k^\omega D_j^{\omega\dagger}] (g_j^\omega | g_k^\omega)_{Q-}^{\omega-} - [D_j^\omega s_t D_k^{\omega\dagger} - s_t D_k^{\omega\dagger} D_j^\omega] \overline{(g_j^\omega | g_k^\omega)_{Q+}^{\omega-}} \}. \end{aligned} \quad (110)$$

From the relation (31) we see that $[Y, H_S] = 0$. If we define the effective evolution operator by

$$V_t = (e^{\frac{t}{i\hbar} H_S} \otimes 1) U_t \quad (111)$$

which satisfies the quantum stochastic differential equation

$$dV_t \equiv (\frac{1}{i\hbar} H_S e^{\frac{t}{i\hbar} H_S} \otimes dt) U_t + (e^{\frac{t}{i\hbar} H_S} \otimes 1) dU_t. \quad (112)$$

Explicitly this gives

$$\begin{aligned} dV_t &\equiv (e^{\frac{t}{i\hbar} H_S} \otimes 1) [\sum_{\omega \in F} \sum_{\phi, \phi' \in B}^{(\omega_{\phi\phi'} = \omega)} \{ T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'}, t) - T_{\phi\phi'}^\dagger \otimes dB_Q^\omega(g_{\phi\phi'}, t) \} - \{ Y + \frac{i}{\hbar} H_S \} dt] U_t \\ &\equiv [\sum_{\omega \in F} \sum_{\phi, \phi' \in B}^{(\omega_{\phi\phi'} = \omega)} \{ T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'}, t) e^{i\omega t} - T_{\phi\phi'}^\dagger \otimes dB_Q^\omega(g_{\phi\phi'}, t) e^{-i\omega t} \} - \{ \frac{1}{2} \Gamma + \frac{i}{\hbar} (H_S + H'_S) \} dt] V_t. \end{aligned}$$

We see that H'_S is a physical addition to the system Hamiltonian due to the presence of the quantum field. This is precisely the Lamb shift. Since Y commutes with H_S it is enough to compute its expectations for eigenstates $\phi \in B$;

$$\langle \phi, Y \phi \rangle = \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ \langle \phi, D_j^{\omega\dagger} D_k^\omega \phi \rangle (g_j^\omega | g_k^\omega)_{Q+}^{\omega-} + \langle \phi, D_j^\omega D_k^{\omega\dagger} \phi \rangle \overline{(g_j^\omega | g_k^\omega)_{Q-}^{\omega-}} \}. \quad (113)$$

This is however equivalent to the (complex) shift one calculates using second order perturbation theory. For example, taking the zero temperature for simplicity, one calculates in second order shift [19]

$$Y_\phi^{(2)} = \frac{1}{i\hbar} \langle \phi \otimes \Psi_R, H_I \frac{1}{H^{(0)} - E_\phi - i0^+} H_I \phi \otimes \Psi_R \rangle \quad (114)$$

$$= \sum_{\omega, \omega' \in F} \sum_j^{N(\omega)} \sum_{j'}^{N(\omega')} \int_{-\infty}^0 d\tau \langle \phi \otimes \Psi_R, D_j^\omega \otimes A(g_{jj'}^\omega) e^{i(H^{(0)} - E_\phi)\tau/\hbar} D_{j'}^{\omega'} \otimes A^\dagger(g_{jj'}^{\omega'}) \phi \otimes \Psi_R \rangle. \quad (115)$$

Here we have used the well known identity

$$\int_{-\infty}^0 dt e^{ixt} = \frac{1}{i(x - i0^+)} = \pi\delta(x) - i\wp(\frac{1}{x}); \quad (x \in \mathbb{R}), \quad (116)$$

where \wp means that we take the principal part of the integral. Now $D_j^\omega \phi$ is an eigenstate of H_S with eigenvalue $E_\phi - \hbar\omega$, so the summation need only be considered over $\omega = \omega'$ in (2.6:17) above. Therefore we have

$$\begin{aligned} Y_\phi^{(2)} &= \sum_{\omega \in F} \sum_{j,k}^{N(\omega)} \int_{-\infty}^0 d\tau \langle \phi \otimes \Psi_R, D_j^{\omega\dagger} \otimes A(g_j^\omega) e^{i(H_R - \hbar\omega)\tau/\hbar} D_k^\omega \otimes A^\dagger(g_k^\omega) \phi \otimes \Psi_R \rangle \\ &= \sum_{\omega \in F} \sum_{j,k}^{N(\omega)} \langle \phi, D_j^{\omega\dagger} D_k^\omega \phi \rangle \int_{-\infty}^0 d\tau \langle \Psi_R, A(g_j^\omega) A^\dagger(S_\tau^\omega g_k^\omega) \Psi_R \rangle. \end{aligned} \quad (117)$$

Hence $Y_\phi^{(2)} = \langle \phi, Y \phi \rangle$. The real and imaginary parts of Y are therefore the linewidth and energy shift as would normally be calculated using second order perturbation theory, this is true in the non-vacuum cases also.

2.7 Transition Probabilities

Let $P_t(\psi|\phi)$ denote the probability that the sytem will be measured in state ψ at time t if it initially was in state ϕ , then

$$p_t(\psi|\phi) = \langle \phi \otimes \Phi_Q^F, U_t^\dagger(|\psi\rangle\langle\psi| \otimes 1) U_t \phi \otimes \Phi_Q^F \rangle, \quad (118)$$

from theorem 3,3a or 3b we have

$$\frac{d}{dt} p_t(\psi|\phi) = \langle \phi \otimes \Phi_Q^F, U_t^\dagger L_0(|\phi\rangle\langle\phi|) \otimes 1 U_t \phi \otimes \Phi_Q^F \rangle. \quad (119)$$

Therefore, setting $t = 0$,

$$\frac{d}{dt} p_t(\psi|\phi)|_{t=0} = \langle \phi, L_0(|\psi\rangle\langle\psi|) \phi \rangle. \quad (120)$$

If $\psi = \phi$ we obtain the relation

$$\frac{d}{dt} p_t(\phi|\phi)|_{t=0} = -\langle \phi, \Gamma \phi \rangle, \quad (121)$$

while if $\langle \phi, \psi \rangle = 0$ then (112) gives

$$\frac{d}{dt} p_t(\psi|\phi)|_{t=0} = \langle \phi, \Theta(|\psi\rangle\langle\psi|) \phi \rangle,$$

$$= \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ \langle \phi, D_j^{\omega\dagger} \psi \rangle \langle \psi, D_k^\omega \phi \rangle (g_j^\omega | g_k^\omega)_{Q+}^\omega + \langle \phi, D_k^\omega \psi \rangle \langle \psi, D_j^{\omega\dagger} \phi \rangle (g_j^\omega | g_k^\omega)_{Q-}^\omega \}, \quad (122)$$

Using the relation $\int_{-\infty}^\infty dt e^{ixt} = 2\pi\delta(x)$ we can rewrite this as

$$\begin{aligned} \frac{d}{dt} p_t(\psi|\phi)|_{t=0} &= 2\pi \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \int dk \overline{g_j^\omega}(\mathbf{k}) g_k^\omega(\mathbf{k}) \delta(\omega(k) - \omega) \\ &\quad \{ \langle \phi, D_j^{\omega\dagger} \psi \rangle \langle \psi, D_k^\omega \phi \rangle \frac{q(\mathbf{k}) + 1}{2} + \langle \phi, D_k^\omega \psi \rangle \langle \psi, D_j^{\omega\dagger} \phi \rangle \frac{q(\mathbf{k}) - 1}{2} \}, \end{aligned} \quad (123)$$

where $q(\mathbf{k})$ is the spectral function associated with Q , cf (125). This is our formulation of the Fermi golden rule for transitions of the systems state and it corresponds to the usual expressions, cf formulae (1.21.27a,b) of [2].

3 THE WEAK COUPLING LIMIT IN QED.

As an illustration of our theory we consider the case of quantum electrodynamics. We stress however that the theory encompasses a wide range of physical phenomena. For instance exciton models in solid state physics such as phonon models or the Frölich [15],[16] polaron model will differ from the following treatment in only minor technical details. The electromagnetic field acts as reservoir for our system (S) which we take to consist of a single electron. The electromagnetic field can be derived from the potential \mathbf{A} given by

$$\mathbf{A}(\mathbf{r}) = \sum_{\sigma=1,2} \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\epsilon_0 c |\mathbf{k}|}} \{ a_\sigma^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} + a_\sigma(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \} \boldsymbol{\epsilon}^\sigma(\hat{\mathbf{k}}). \quad (124)$$

Here we consider two transverse polarizations ($\sigma = 1, 2$) for each mode \mathbf{k} . In our notation $\{\epsilon^1(\hat{\mathbf{k}}), \epsilon^2(\hat{\mathbf{k}}), \hat{\mathbf{k}} = |\mathbf{k}|^{-1}\mathbf{k}\}$ form a right-handed triad for each \mathbf{k} . This ensures that we are working with the radiation gauge $\nabla \cdot \mathbf{A} = 0$. The operators $a_\sigma^\dagger(\mathbf{k})$ on the reservoir state space \mathcal{H}_R satisfy Bose commutation relations

$$[a_\sigma(\mathbf{k}), a_{\sigma'}^\dagger(\mathbf{k}')] = \delta_{\sigma\sigma'} \delta(\mathbf{k} - \mathbf{k}'). \quad (125)$$

The total Hamiltonian for the system and reservoir is

$$\begin{aligned} H &= \frac{1}{2m} |\mathbf{p} - e\mathbf{A}|^2 + \Phi(\mathbf{r}) + H_R \\ &= H_S + H_R + H_I + H'_I, \end{aligned} \quad (126)$$

where the unperturbed system Hamiltonian (with potential $\Phi(\mathbf{r})$) is

$$H_S = \frac{1}{2m} |\mathbf{p}|^2 + \Phi(\mathbf{r}); \quad (127)$$

$$H_R = \sum_{\sigma=1,2} \int d^3k \, \hbar c |k| \, a_\sigma^\dagger(\mathbf{k}) a_\sigma(\mathbf{k}), \quad (128)$$

$$H_I = -\frac{e}{m} \sum_{\sigma=1,2} \int d^3k \, \{a_\sigma^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{r}} + a_\sigma(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}\} \mathbf{G}^\sigma(\mathbf{k}) \cdot \mathbf{p}, \quad (129)$$

with

$$\mathbf{G}^\sigma(\hat{\mathbf{k}}) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\epsilon_0 c |\mathbf{k}|}} \epsilon^\sigma(\hat{\mathbf{k}});$$

and

$$H'_I = \frac{e^2}{2m} |\mathbf{A}|^2.$$

If we rescale the electronic charge as $e \mapsto \lambda e$ we find that

$$H \mapsto H_S + H_R + \lambda H_I + \lambda^2 H'_I.$$

In the subsequent analysis we shall drop the term $\lambda^2 H'_I$ and consider only

$$H^{(\lambda)} = H_S + H_R + \lambda H_I. \quad (130)$$

It has been established rigorously that this does not effect the final result in the weak coupling limit. Now the interaction H_I given by (120) has response terms described by the vectors $\theta_j^\sigma(\mathbf{k}) = \frac{ie}{\hbar m} e^{-i\mathbf{k} \cdot \mathbf{r}} G_j^\sigma(\hat{\mathbf{k}}) \cdot \mathbf{p}$. We assume, as usual, that the unperturbed system Hamiltonian H_S has a complete orthonormal set of eigenstates B . In the case of the Hydrogen atom, this means that we consider only the bound states and ignore the effect of the ionized states. In general, \mathcal{H}_S can be decomposed into complementary subspaces generated by the discrete, the absolutely continuous and the singular parts of the spectrum of H_S . It is enough to prepare the system in the discrete spectrum subspace to apply our results. It is the standard approach in atomic physics to study only the behaviour of bound states anyway, so we are justified in this restriction.

We introduce the test-functions

$$g_{\phi\phi'}^\sigma(\mathbf{k}) = \frac{ie}{\hbar m} \langle \phi | e^{-i\mathbf{k} \cdot \mathbf{r}} \mathbf{p} | \phi' \rangle \cdot \mathbf{G}^\sigma(\hat{\mathbf{k}}). \quad (131)$$

The interaction H_I can be expressed as

$$\begin{aligned} H_I &= \sum_{\phi, \phi' \in B} \sum_{\sigma=1,2} \int d^3k \, \{a_\sigma^\dagger(\mathbf{k}) g_{\phi\phi'}^\sigma(\mathbf{k}) - a_\sigma(\mathbf{k}) \bar{g}_{\phi\phi'}^\sigma(\mathbf{k})\} \otimes T_{\phi\phi'} \\ &= \sum_{\phi, \phi' \in B} \sum_{\sigma=1,2} \int d^3k \, \{T_{\phi\phi'} \otimes a_\sigma^\dagger(\mathbf{k}) g_{\phi\phi'}^\sigma(\mathbf{k}) - T_{\phi'\phi} \otimes a_\sigma(\mathbf{k}) \bar{g}_{\phi\phi'}^\sigma(\mathbf{k})\} \\ &= \sum_{\phi, \phi' \in B} \{T_{\phi\phi'} \otimes A^\dagger(g_{\phi\phi'}) - T_{\phi\phi'}^\dagger \otimes A(g_{\phi\phi'})\}, \end{aligned} \quad (132)$$

where $g_{\phi\phi'} = g_{\phi\phi'}^1 \oplus g_{\phi\phi'}^2 \in L^2(R^3) \oplus L^2(R^3) = \mathcal{H}_R^1$ and A^\dagger are the creation/ annihilation operators on $\Gamma_B(\mathcal{H}_R^1) = \otimes \Gamma_B(L^2(\mathbb{R}))$;

$$\begin{aligned} A^\dagger(f^1 \oplus f^2) &= \sum_{\sigma=1,2} \int d^3k f^\sigma(\mathbf{k}) a_\sigma^\dagger(\mathbf{k}), \\ A(f^1 \oplus f^2) &= \sum_{\sigma=1,2} \int d^3k \bar{f}^\sigma(\mathbf{k}) a_\sigma(\mathbf{k}). \end{aligned} \quad (133)$$

Our choice of \mathcal{H}_R^1 above for one particle of the reservoir space is quite natural; namely it consists of wave-functions in the momentum representation with two transverse polarizations. The state of the reservoir is in our case determined by the covariance operator Q which we shall now specify as that of a thermal field at inverse temperature β 0, that is

$$\begin{aligned} Q : \mathcal{H}_R^1 &\mapsto \mathcal{H}_R^1 : h_1 \oplus h_2 \mapsto \tilde{h}_1 \oplus \tilde{h}_2 \\ \text{with } \tilde{h}_\sigma(\mathbf{k}) &= q(c|\mathbf{k}|) h_\sigma(\mathbf{k}), \end{aligned} \quad (134)$$

where $q(\omega) = \coth \frac{\beta \hbar \omega}{2}$. With $\omega_{\phi\phi'} = (E_{\phi'} - E_\phi)/\hbar \in F$, we define

$$S_t^{\omega_{\phi\phi'}} : \mathcal{H}_R^1 \mapsto \mathcal{H}_R^1 : h_1 \oplus h_2 \mapsto \tilde{h}_1 \oplus \tilde{h}_2$$

with

$$\tilde{h}_\sigma(\mathbf{k}) = e^{i(c|\mathbf{k}| - \omega_{\phi\phi'})t} h_\sigma(\mathbf{k}). \quad (135)$$

In this setup we have allowed for the most general coupling, that is where all the fundamental frequencies $F = \{\omega_{\phi\phi'} : \phi, \phi' \in B\}$ are to be considered. This set is always degenerate in general; however it is important to consider two classes of degeneracy arising. The first is the secular class; these are the situations in which degeneracies always arise regardless of the spectrum $\{E_\phi : \phi \in B\}$ of H_S ; they are the pairs (ϕ, ϕ') and (ψ, ψ') which have $\omega_{\phi\phi'} = \omega_{\psi\psi'}$ due to one of the following reasons

1. $\phi = \phi' = \psi = \psi'$,
 2. $\phi = \phi', \psi = \psi'; (\phi \neq \psi)$,
 3. $\phi = \psi, \phi' = \psi'; (\phi \neq \phi')$.
- (136)

Any solution to the equation $\omega_{\phi\phi'} = \omega_{\psi\psi'}$, or equivalently $E_\phi - E_{\phi'} = E_\psi - E_{\psi'}$, not of the secular type shall be called an extraneous solution. The extraneous solutions are of course dependent on the spectrum of H_S . It is standard procedure in the physical literature to assume that such possibilities do not arise however this is a requirement on H_S which cannot be made in many important examples. For a particle in a rectangular box, apart from the natural degeneracies arising if the ratios of the sides are rational, we also have to consider the fact that the contribution to the energy for the mode of vibration n_i along the i^{th} -axis is proportional to n_i^2 , this means solving the Diophantine equations for the harmonics

$$n_i^2 - m_i^2 = n_i'^2 - m_i'^2.$$

For the Hydrogen atom we have, apart from the spherical harmonical degeneracies, to consider the integer solutions to the Diophantine equations

$$\frac{1}{n^2} - \frac{1}{m^2} = \frac{1}{n'^2} - \frac{1}{m'^2}$$

for the principal atomic numbers. After simple manipulations this leads to the study of the intersection of the algebraic projective curve in \mathbb{R}^4 ;

$$x_1^2 x_3^2 x_4^2 - x_2^2 x_3^2 x_4^2 - x_1^2 x_2^2 x_4^2 + x_1^2 x_2^2 x_3^2 = 0$$

with the lattice of positive integers. In the weak coupling limit we obtain the quantum stochastic differential equation

$$dU_t = \left[\sum_{\omega \in F} \sum_{\phi, \phi' \in B}^{(\omega_{\phi\phi'} = \omega)} \{T_{\phi\phi'} \otimes dB_Q^{\omega\dagger}(g_{\phi\phi'}, t) + T_{\phi\phi'}^\dagger \otimes dB_Q^\omega(g_{\phi\phi'}, t)\} + Y dt \right] U_t$$

where

$$Y = \sum_{\omega \in F} \sum_{\phi, \phi', \psi, \psi' \in B}^{(\omega_{\phi\phi'} = \omega = \omega_{\psi\psi'})} [T_{\phi\phi'}^\dagger T_{\psi\psi'}(g_{\phi\phi'} | g_{\psi\psi'})_{Q+}^{\omega-} + T_{\phi\phi'} T_{\psi\psi'}^\dagger \overline{(g_{\phi\phi'} | g_{\psi\psi'})_{Q-}^{\omega-}}]; \quad (137)$$

with $U_0 = 1$. However using the fact that $T_{\phi\phi'}^\dagger T_{\psi\psi'} = \langle\phi, \psi\rangle T_{\phi'\psi'}$ etc., we may write Y as

$$Y = \sum_{\omega \in F} \sum_{\phi, \psi, \phi' \in B}^{(\omega = \omega_{\phi\phi'} = \omega_{\psi\psi'})} \{ (g_{\phi'\phi} | g_{\phi'\psi})_{Q+}^{(-\omega)-} + \overline{(g_{\phi\phi'} | g_{\psi\phi'})_{Q-}^{\omega-}} \} T_{\phi\psi}. \quad (138)$$

In the summation we consider only ϕ and ψ for which there exists a ϕ' so that $\omega_{\phi\phi'} = \omega_{\psi\psi'}$, however this is equivalent to demanding that $\omega_{\phi\psi} = 0$ as we always have the identity $\omega_{\phi\psi} = \omega_{\phi\phi'} - \omega_{\psi\phi'}$. Therefore Y is a linear combination of terms $T_{\phi\psi}$ with $\omega_{\phi\psi} = 0$ and this, in particular, implies that Y commutes with H_S . It is natural in light of this to write Y as

$$Y = \sum_{\phi, \psi \in B}^{(\omega_{\phi\psi}=0)} y_{\phi\psi} T_{\phi\psi} = \sum_{\phi, \psi \in B}^{(E_\phi=E_\psi)} y_{\phi\psi} T_{\phi\psi}; \quad (139)$$

where

$$y_{\phi\psi} = \sum_{\phi' \in B} \{ (g_{\phi'\phi} | g_{\phi'\psi})_{Q+}^{\omega_{\phi'\phi}-} + \overline{(g_{\phi\phi'} | g_{\psi\phi'})_{Q-}^{\omega_{\phi\phi'}-}} \}. \quad (140)$$

Now the master equation associated with this problem is from (105)

$$\frac{ds_t}{dt} = L_0^*(s_t) = -(Y s_t + s_t Y^\dagger) + \sum_{\omega \in F} \sum_{\phi, \phi', \psi, \psi' \in B}^{(\omega = \omega_{\phi\phi'} = \omega_{\psi\psi'})} [T_{\phi\phi'}^\dagger s_t T_{\psi\psi'} (g_{\phi\phi'} | g_{\psi\psi'})_{Q+}^\omega + T_{\phi\phi'} s_t T_{\psi\psi'}^\dagger (g_{\psi\psi'} | g_{\phi\phi'})_{Q-}^\omega]. \quad (141)$$

In order to find the general expression for H'_S , we see that

$$\begin{aligned} y_{\phi\psi} &= \sum_{\phi' \in B} \int_{-\infty}^0 d\tau \{ \langle g_{\phi'\phi}, S_\tau^{\omega_{\phi'\phi}} \frac{Q+1}{2} g_{\phi'\psi} \rangle + \overline{\langle g_{\phi\phi'}, S_\tau^{\omega_{\phi\phi'}} \frac{Q-1}{2} g_{\psi\phi'} \rangle} \} \\ &= \sum_{\phi' \in B} \int_{-\infty}^0 d\tau \sum_{\sigma=1,2} \int d^3k \{ \overline{g_{\phi'\phi}^\sigma}(\mathbf{k}) g_{\phi'\psi}^\sigma(\mathbf{k}) e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + g_{\phi\phi'}^\sigma(\mathbf{k}) \overline{g_{\psi\phi'}^\sigma}(\mathbf{k}) e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|-1)}{2} \} e^{i\omega_{\phi'\phi}t}. \end{aligned}$$

But using $\overline{g_{\phi\phi'}^\sigma}(\mathbf{k}) = -g_{\phi'\phi}^\sigma(-\mathbf{k})$ we have

$$\begin{aligned} y_{\phi\psi} &= \sum_{\phi' \in B} \int_{-\infty}^0 d\tau \sum_{\sigma=1,2} \int d^3k g_{\phi\phi'}^\sigma(\mathbf{k}) \overline{g_{\psi\phi'}^\sigma}(\mathbf{k}) \{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|-1)}{2} \} e^{i\omega_{\phi'\phi}t} \\ &= \frac{e^2}{\hbar^2 m^2} \sum_{\phi' \in B} \int_{-\infty}^0 d\tau \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | e^{-i\mathbf{k}\cdot\mathbf{r}} p_j | \phi' \rangle \langle \phi' | e^{i\mathbf{k}\cdot\mathbf{r}} p_{j'} | \psi \rangle \\ &\quad \times G_j^\sigma(\mathbf{k}) G_{j'}^\sigma(\mathbf{k}) \{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|-1)}{2} \} e^{i\omega_{\phi'\phi}t} \\ &= \frac{e^2}{\hbar^2 m^2} \sum_{\phi' \in B} \int_{-\infty}^0 d\tau \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | e^{\frac{t}{i\hbar} E_\phi} e^{-i\mathbf{k}\cdot\mathbf{r}} p_j e^{\frac{-t}{i\hbar} H_S} e^{i\mathbf{k}\cdot\mathbf{r}} p_{j'} | \psi \rangle \\ &\quad \times G_j^\sigma(\mathbf{k}) G_{j'}^\sigma(\mathbf{k}) \{ e^{-ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|)+1}{2} + e^{ic|\mathbf{k}|t} \frac{q(c|\mathbf{k}|-1)}{2} \}. \end{aligned} \quad (142)$$

We remark that the effect of the response term is as follows; from the commutation relations of \mathbf{r} and \mathbf{p} we have that

$$e^{-i\mathbf{k}\cdot\mathbf{r}} e^{\frac{t}{i\hbar} H_S} e^{i\mathbf{k}\cdot\mathbf{r}} = e^{-i\mathbf{k}\cdot\mathbf{r}} \exp \frac{t}{i\hbar} \left(\frac{|\mathbf{p}|^2}{2m} + V(r) \right) e^{i\mathbf{k}\cdot\mathbf{r}} = \exp \frac{t}{i\hbar} \left(\frac{|\mathbf{p} + \hbar\mathbf{k}|^2}{2m} + V(r) \right), \quad (143)$$

that is H_S is replaced by $H_S + \frac{\hbar\mathbf{k}\cdot\mathbf{p}}{m} + \frac{\hbar^2|\mathbf{k}|^2}{2m}$. Now the \mathbf{k} dependence in the above expression prevents us from using the well-known isotropic identity

$$\sum_{\sigma=1,2} \int_{|\mathbf{k}|=\omega/c} d^2\hat{k} G_j^\sigma(\mathbf{k}) G_{j'}^\sigma(\mathbf{k}) = \frac{1}{(2\pi)^3} \frac{\hbar}{2\epsilon_0\omega} \frac{8\pi}{3} \delta_{j,j'}, \quad (144)$$

to calculate $y_{\phi\psi}$ as in the dipole approximation. Note that the Lamb shift and the damping coefficients are effected by inclusion of the response terms.

The complex shift $Y_{\phi\phi} = \frac{1}{2}\Gamma_\phi + \frac{i}{\hbar}E'_\phi$, giving the linewidth Γ_ϕ and energy shift E'_ϕ for a state $\phi \in B$ can be written as

$$Y_{\phi\phi} = \frac{e^2}{2i\hbar m^2} \sum_{\sigma=1,2} \int d^3k \sum_{j,j'=1,2,3} \langle \phi | p_j [\frac{q(c|\mathbf{k}|) + 1}{\mathcal{D}^+(\mathbf{k}) - i0^+} + \frac{q(c|\mathbf{k}|) - 1}{\mathcal{D}^-(\mathbf{k}) - i0^+}] p_{j'} | \phi \rangle G_j^\sigma(\mathbf{k}) G_{j'}^\sigma(\mathbf{k}), \quad (145)$$

where the denominators in the above expression are

$$\mathcal{D}^\pm(\mathbf{k}) = H_S + \frac{\hbar \mathbf{k} \cdot \mathbf{p}}{m} + \frac{\hbar^2 |\mathbf{k}|^2}{2m} \pm \hbar c |\mathbf{k}| - E_\phi. \quad (146)$$

This expression has been derived in [20] for the zero temperature case; see however [21].

APPENDIX. A: The Traditional Derivation of the Master Equation.

For the sake of comparison we give the standard arguments used in the derivation of the master equation. This section follows closely the development of Louisell [2]. The interaction is taken to be of the form

$$H_I = \sum_j D_j \otimes F_j,$$

where D_j and F_j act nontrivially on the system and reservoir spaces respectively. We assume that D_j evolves harmonically in time under the free evolution with frequency ω_j . We assume that at time t the system and reservoir are uncoupled, that is the density operator $\rho(t)$ at time zero factors as

$$\rho(0) = \rho_0^{(S)} \otimes \rho^{(R)}.$$

No subscript is required for $\rho^{(R)}$ as we assume that it is invariant under the free-evolution (in particular this is true for the choice of a thermal state $\rho^{(R)} = e^{-\beta(H_R^1 - \mu)} / \text{Tr} e^{-\beta(H_R^1 - \mu)}$). We define the reduced system state at time t in the interaction dynamics to be the density operator

$$s_t = \text{Tr}_{\mathcal{H}_R} \{ U_t^{(\lambda)} (\rho_0^{(S)} \otimes \rho^{(R)}) U_t^{(\lambda)\dagger} \}.$$

The iterated series expansion of s_t , truncated to second order, is;

$$s_t = s_0 + \frac{1}{i\hbar} \int_0^t dt_1 \text{Tr}_{\mathcal{H}_R} [v_{t_1}^{(0)}(H_I), s_0 \otimes \rho^{(R)}] + \frac{1}{(i\hbar)^2} \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Tr}_{\mathcal{H}_R} [v_{t_1}^{(0)}(H_I), [v_{t_2}^{(0)}(H_I), s_0 \otimes \rho^{(R)}]],$$

where we have set $\lambda = 1$. Substituting in for the potential H_I we find

$$\begin{aligned} s_t &= s_0 + \sum_j \int_0^t dt_1 \langle v_{t_1}^{(0)}(F_j) \rangle_R e^{-i\omega_j t_1} [D_j, s_0] \\ &+ \sum_{j,k} \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-i(\omega_j t_1 + \omega_k t_2)} \{ [D_j D_k s_0 - D_k s_0 D_j] \langle v_{t_1}^{(0)}(F_j) v_{t_2}^{(0)}(F_k) \rangle_R - [D_j s_0 D_k - s_0 D_k D_j] \langle v_{t_2}^{(0)}(F_k) v_{t_1}^{(0)}(F_j) \rangle_R \}, \end{aligned}$$

where $\langle \cdot \rangle_R = \text{Tr}_{\mathcal{H}_R} [\rho^{(R)}]$. Due to the invariance of the reservoir fields under the free evolution we have that

$$\langle v_t^{(0)}(F_j) \rangle_R = \langle F_j \rangle_R;$$

$$\langle v_t^{(0)}(F_j) v_s^{(0)}(F_k) \rangle_R = \langle v_{t-s}^{(0)}(F_j) F_k \rangle_R.$$

Therefore, if we let $\tau = t_1 - t_2, y = t_2$ then we obtain

$$\begin{aligned} s_t &= s_0 + \sum_j \langle F_j \rangle_R [D_j, s_0] \int_0^t e^{-i\omega_j y} dy \\ &+ \sum_{j,k} \int_0^t dy e^{-i(\omega_j + \omega_k)y} \int_0^{t-y} d\tau e^{-i\omega_j \tau} \{ [D_j D_k s_0 - D_k s_0 D_j] \langle v_\tau^{(0)}(F_j) F_k \rangle_R - [D_j s_0 D_k - s_0 D_k D_j] \langle F_k v_\tau^{(0)}(F_j) \rangle_R \}. \end{aligned}$$

The approximation procedure is based on the following four steps; *step I*. One postulates that the contributions coming from the sum of all terms higher than second order in the iterated series are negligible. *step II*. One postulates a finite autocorrelation time τ_c such that

$$\langle v_\tau^{(0)}(F_j)F_k \rangle_R = 0 = \langle F_j v_\tau^{(0)}(F_k) \rangle_R,$$

whenever $|\tau| > \tau_c$. Thus for $t > \tau_c$ one may replace the upper limit of the τ -integral by $+\infty$. This gives

$$s_t = s_0 + \sum_j \langle F_j \rangle_R [D_j, s_0] I^t(\omega_j) \\ + \sum_{j,k} \{ [D_j D_k s_0 - D_k s_0 D_j] w_{j,k}^+ - [D_j s_0 D_k - s_0 D_k D_j] w_{k,j}^- \} I^t(\omega_j + \omega_k),$$

where

$$w_{j,k}^+ = \int_0^\infty e^{-i\omega_j \tau} \langle v_\tau^{(0)}(F_j)F_k \rangle_R d\tau \\ w_{k,j}^- = \int_0^\infty e^{-i\omega_j \tau} \langle F_k v_\tau^{(0)}(F_j) \rangle_R d\tau$$

and

$$I^t(\omega) = \int_0^t e^{-i\omega y} dy.$$

Step III. For t large with respect to τ_c one makes the replacement

$$I^t(\omega) \hookrightarrow t\delta(\omega).$$

Step IV. One postulates that the formulas deduced under the previous assumptions when t is large with respect to τ_c hold also in the limit $t \rightarrow 0$; this gives

$$\frac{ds}{dt}|_0 = \lim_{t \rightarrow 0} \frac{s_t - s_0}{t} \\ = \sum_j^{\omega_j=0} [D_j, s_0] \langle F_j \rangle_R + \sum_{j,k}^{\omega_j+\omega_k=0} \{ [D_j D_k s_0 - D_k s_0 D_j] w_{j,k}^+ - [D_j s_0 D_k - s_0 D_k D_j] w_{k,j}^- \}.$$

The assumptions leading to this equation have a decidedly ad hoc nature, especially those introduced in steps III and IV. The replacement for I^t , put in by hand, in step III is precisely what is needed to allow the limit to be taken easily. It is instructive to calculate explicitly the master equation in a particular case. We consider as reservoir a free Bose gas at inverse temperature β and fugacity $z = e^{\beta\mu}$. This can be described by the quasi-free state φ_Q , on $L^2(\mathbb{R}^n)$ for example, characterized by

$$\langle A^\dagger(f)A(g) \rangle_R \equiv \varphi_Q(A^\dagger(f)A(g)) = \langle f, \frac{Q-1}{2}g \rangle,$$

where

$$Q = \frac{1 + ze^{-\beta H_R^1}}{1 - ze^{-\beta H_R^1}} = \coth \frac{\beta}{2}(H_R^1 - \mu).$$

We may take H_R^1 to be for instance $-\Delta$.

We may write the interaction H_I of (31) in the form we are considering with the notations

$$H_I = i\hbar \sum_j \{ D_j \otimes A^\dagger(g_j) - h.c. \} \equiv i\hbar \sum_{(j,\alpha)} D_{(j,\alpha)} \otimes F_{(j,\alpha)},$$

where we have a summation also over an index $\alpha \in \{0, 1\}$ with the notations

$$D_{(j,0)} = D_j, \quad D_{(j,1)} = -D_j^\dagger; \quad F_{(j,0)} = A^\dagger(g_j), \quad F_{(j,1)} = A(g_j);$$

and consequently

$$\omega_{(j,0)} = \omega_j; \quad \omega_{(j,1)} = -\omega_j.$$

We then have

$$\begin{aligned} w_{(j0),(k1)}^+ &= \int_0^\infty d\tau e^{-i\omega_j \tau} \langle v_\tau^{(0)} (A^\dagger(g_j)) A(g_k) \rangle_R = \int_0^\infty d\tau \varphi_Q(A^\dagger(S_\tau^{\omega_j} g_j) A(g_k)) \\ &= \int_0^\infty d\tau \langle S_\tau^{\omega_j} g_j, \frac{Q-1}{2} g_k \rangle = \overline{(g_j|g_k)}_{Q-}^{\omega_j-}, \end{aligned}$$

Similarly, using the CCR, we find

$$\begin{aligned} w_{(j1),(k0)}^+ &= \int_0^\infty d\tau e^{i\omega_j \tau} \langle v_\tau^{(0)} (A(g_j)) A^\dagger(g_k) \rangle_R = (g_j|g_k)_{Q+}^{\omega_j-}, \\ w_{(k0),(j1)}^- &= (g_j|g_k)_{Q-}^{\omega_j-}, \\ w_{(j1),(k0)}^- &= \overline{(g_j|g_k)}_{Q+}^{\omega_j-}, \end{aligned}$$

while $w_{(j\epsilon),(j'\epsilon')}^\pm = 0$ if $\epsilon = \epsilon'$ as we have $\langle A(f)A(g) \rangle_R = 0 = \langle A^\dagger(f)A^\dagger(g) \rangle_R$. We note that $\langle F_{(j,\alpha)} \rangle_R = 0$ in all cases. The master equation then reads

$$\begin{aligned} \frac{ds}{dt}|_0 &= \sum_{j,k;\alpha,\alpha'}^{(-1)^{\alpha}\omega_j + (-1)^{\alpha'}\omega_k = 0} \{ [D_{j,\alpha} D_{k,\alpha'} s_0 - D_{k,\alpha'} s_0 D_{j,\alpha}] w_{(j\alpha),(k\alpha')}^+ \\ &\quad - [D_{j,\alpha} s_0 D_{k,\alpha'} - s_0 D_{k,\alpha'} D_{j,\alpha}] w_{(k\alpha'),(j\alpha)}^- \} \\ &= - \sum_{j,k}^{\omega_j - \omega_k = 0} \{ [D_j D_k^\dagger s_0 - D_k^\dagger s_0 D_j] w_{(j0),(k1)}^+ + [D_j^\dagger D_k s_0 - D_k s_0 D_j^\dagger] w_{(j1),(k0)}^+ \\ &\quad - [D_j^\dagger s_0 D_k - s_0 D_k D_j^\dagger] w_{(k0),(j1)}^- - [D_j s_0 D_k^\dagger - s_0 D_k^\dagger D_j] w_{(k1),(j0)}^- \}, \end{aligned}$$

or writing in our notations (and employing the relabeling in terms of the frequency degeneracies as in (31))

$$\begin{aligned} \frac{ds}{dt}|_0 &= - \sum_{\omega \in F} \sum_{j,k=1}^{N(\omega)} \{ [D_j^\omega D_k^{\omega\dagger} s_0 - D_k^{\omega\dagger} s_0 D_j^\omega] \overline{(g_j^\omega|g_k^\omega)}_{Q-}^{\omega-} \\ &\quad + [D_j^{\omega\dagger} D_k^\omega s_0 - D_k^\omega s_0 D_j^{\omega\dagger}] (g_j^\omega|g_k^\omega)_{Q+}^{\omega-} \\ &\quad - [D_j^{\omega\dagger} s_0 D_k^\omega - s_0 D_k^\omega D_j^{\omega\dagger}] (g_j^\omega|g_k^\omega)_{Q-}^{\omega-} - [D_j^\omega s_0 D_k^{\omega\dagger} - s_0 D_k^{\omega\dagger} D_j^\omega] \overline{(g_j^\omega|g_k^\omega)}_{Q+}^{\omega-} \}. \end{aligned}$$

But this is exactly $\frac{ds_t}{dt}|_0 = L_0^*(s_0)$, where L_0^* is given as before.

APPENDIX B: The Convergence of the Collective Processes to the Noise Processes

The mathematical theory behind the weak coupling limit developed in [1] and subsequent papers is the following. We estimate the behaviour as $\lambda \rightarrow 0$ of matrix elements of $U_{t/\lambda^2}^{(\lambda)}$ with respect to collective coherent vectors, that is vectors of the form $B_{t^{(1)}/\lambda^2}^{(\omega,\lambda)}(f^{(1)}) \dots B_{t^{(n)}/\lambda^2}^{(\omega,\lambda)}(f^{(n)}) \Psi_R$. This involves substituting $v_t(H_I)$ into the series expansion for $U_{t/\lambda^2}^{(\lambda)}$ and examining each of the terms arising. The deep analysis of [1] shows that each term upon normal ordering leads to two classes of terms: relevant ones (type I) and negligible ones (type II). The type I terms are exactly those put into normal order by commuting *time consecutive* pairs of reservoir variables, the type II terms account for all others. Following this resummation it is shown that the type II terms give vanishing contribution in the limit $\lambda \rightarrow 0$ while the explicit limit for the type I terms is calculated; uniform convergence is established, the main technical device used here is one of various generalizations of the Pulé inequality [3].

The independence of the noise processes for different frequencies follows from the next two lemmas.

Lemma 1: *for each $\omega \in F$ let $f_\omega, f'_\omega \in K_\omega$ and $S_\omega, T_\omega, S'_\omega, T'_\omega \in R$ then*

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \langle &\lambda \int_{S_\omega/\lambda^2}^{T_\omega/\lambda^2} S_u^\omega f_\omega du, \lambda \int_{S'_\omega/\lambda^2}^{T'_\omega/\lambda^2} S_v^\omega f'_\omega dv \rangle \\ &= \sum_{\omega \in F} \langle \chi_{[S_\omega, T_\omega]}, \chi_{[S'_\omega, T'_\omega]} \rangle_{L^2(\mathbb{R})} (f_\omega | f'_\omega)_\omega \\ &= (\oplus_{\omega \in F} (\chi_{[S_\omega, T_\omega]} \otimes f_\omega) | \oplus_{\omega' \in F} (\chi_{[S'_{\omega'}, T'_{\omega'}]} \otimes f'_{\omega'})). \end{aligned}$$

Proof. the left hand side of (B.1) can be written as a sum over $\omega, \omega' \in F$ of terms

$$\lim_{\lambda \rightarrow 0} \int_{S_\omega}^{T_\omega} du \int_{(S'_{\omega'} - u)/\lambda^2}^{(T'_{\omega'} - u)/\lambda^2} du' \langle f_\omega, S_{u'}^\omega f'_{\omega'} \rangle e^{i(\omega - \omega')u/\lambda^2};$$

by the Riemann-Lesbegue Lemma the terms $\omega \neq \omega'$ vanish while the $\omega = \omega'$ terms converge by inspection to

$$\langle \chi_{[S_\omega, T_\omega]}, \chi_{[S'_{\omega'}, T'_{\omega'}]} \rangle_{L^2(\mathbb{R})} (f_\omega | f'_{\omega'})_\omega$$

■

Lemma 2: for $n \in N$ let $f_\omega^{(k)} \in K_\omega, x_\omega^{(k)} \in R, S_\omega^{(k)} \langle T_\omega^{(k)} \rangle$, for $1 \leq k \leq n$ and each $\omega \in F$, then

$$\lim_{\lambda \rightarrow 0} \langle \Phi_Q^F, B_Q^{F\dagger} \left(\sum_{\omega \in F} x_\omega^{(1)} \lambda \int_{S_\omega^{(1)}}^{T_\omega^{(1)}} S_{u_1}^\omega f_\omega^{(1)} du_1 \right) \dots B_Q^{F\dagger} \left(\sum_{\omega \in F} x_\omega^{(n)} \lambda \int_{S_\omega^{(n)}}^{T_\omega^{(n)}} S_{u_n}^\omega f_\omega^{(n)} du_n \right) \Phi_Q^F \rangle$$

exists uniformly for the x 's and $[S, T]$'s in a bounded set of R and is equal to

$$\langle \Phi_R^Q, W(\oplus_{\omega \in F} (x_\omega^{(1)} \chi_{[S_\omega^{(1)}, T_\omega^{(1)}]} \otimes f_\omega^{(1)})) \dots W(\oplus_{\omega \in F} (x_\omega^{(n)} \chi_{[S_\omega^{(n)}, T_\omega^{(n)}]} \otimes f_\omega^{(n)})) \Phi_R^Q \rangle.$$

For a proof see [10].

APPENDIX C: The Quantum Stochastic Differential Equation for U_t .

For convenience we shall consider only one coupling frequency ω so that

$$H_I \equiv i\hbar \sum_j (D_j \otimes A^\dagger(g_j))$$

and we have dropped the superscript ω from the operators. Also we shall consider only the Fock (vacuum) case $Q = 1$. We define $G_\lambda(t)$ by

$$\langle \psi, G_\lambda(t) \rangle = \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega, \lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega, \lambda)}(f) U_{t/\lambda^2}^{(\lambda)}\},$$

where $\psi, \phi \in \mathcal{H}_S$ and the second inner product is meant on $\mathcal{H}_S \otimes \Gamma_B(L^2(\mathbb{R}, K_\omega))$: we know that the limit $\lim_{\lambda \rightarrow 0} \langle \psi, G_\lambda(t) \rangle$ exists and equals

$$\langle \psi \otimes B_Q^{\omega\dagger}(f, T) \Phi_Q^\omega, U_t \phi \otimes B_Q^{\omega\dagger}(f', T') \Phi_Q^\omega \rangle.$$

It is easy to show that this limit has the form $\langle \psi, G(t) \rangle$ where $: t \mapsto G(t) \in \mathcal{H}_S$ is weakly differentiable. In order to obtain a differential equation for $G(t)$ we note that for fixed λ one has

$$\begin{aligned} \frac{d}{dt} \langle \psi, G_\lambda(t) \rangle &= \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega, \lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega, \lambda)}(f) \frac{1}{\lambda} \sum_j (D_j \otimes A^\dagger(S_{t/\lambda^2}^\omega g_j) - D_j^\dagger \otimes A(S_{t/\lambda^2}^\omega g_j)) U_{t/\lambda^2}^{(\lambda)}\} \\ &= \Gamma_\lambda + \Xi_\lambda; \end{aligned}$$

where

$$\begin{aligned} \Gamma_\lambda &= \frac{1}{\lambda} \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega, \lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega, \lambda)}(f) \sum_j D_j \otimes A^\dagger(S_{t/\lambda^2}^\omega g_j) U_{t/\lambda^2}^{(\lambda)}\}, \\ \Xi_\lambda &= -\frac{1}{\lambda} \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega, \lambda)\dagger}(f') |\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega, \lambda)}(f) \sum_j D_j^\dagger \otimes A(S_{t/\lambda^2}^\omega g_j) U_{t/\lambda^2}^{(\lambda)}\}. \end{aligned}$$

Now

$$\begin{aligned}\Gamma_\lambda &= \sum_j \frac{1}{\lambda} \int_0^{T/\lambda^2} \langle S_u^\omega f, S_{t/\lambda^2}^\omega g_j \rangle du \langle D_j^\dagger \psi, G_\lambda(t) \rangle \\ &= \sum_j \frac{1}{\lambda} \int_{(-t)/\lambda^2}^{(T-t)/\lambda^2} \langle S_v^\omega f, g_j \rangle dv \langle D_j^\dagger \psi, G_\lambda(t) \rangle\end{aligned}$$

where we made the substitution $u - t/\lambda^2 = v$. We see that for bounded D this converges as $\lambda \rightarrow 0$ a.e. to

$$\sum_j \chi_{[S,T]}(f|g_j)_\omega \langle D_j^\dagger \psi, G_\lambda(t) \rangle.$$

Next of all the term Ξ_λ must be reordered as follows

$$\begin{aligned}\Xi_\lambda &= -\frac{1}{\lambda} \sum_j \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega,\lambda)\dagger}(f')|\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega,\lambda)}(f) (D_j^\dagger \otimes 1) U_{t/\lambda}^{(\lambda)}(1 \otimes A(S_{t/\lambda^2}^\omega))\} \\ &\quad -\frac{1}{\lambda} \sum_j \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega,\lambda)\dagger}(f')|\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega,\lambda)}(f) (D_j^\dagger \otimes 1) [(1 \otimes A(S_{t/\lambda^2}^\omega)), U_{t/\lambda}^{(\lambda)}]\} \\ &= \Xi_\lambda^a + \Xi_\lambda^b.\end{aligned}$$

In a fashion similar to the calculation of Γ_λ , one easily arrives at

$$\lim_{\lambda \rightarrow 0} \Xi_\lambda^a = - \sum_j \chi_{[0,T']}(g_j|f')_\omega \langle D_j^\dagger \psi, G_\lambda(t) \rangle, \text{ a.e.}$$

To evaluate the limit of Ξ_λ^b we note that

$$\begin{aligned}[(1 \otimes A(S_{t/\lambda^2}^\omega g_j)), U_{t/\lambda^2}^{(\lambda)}] &= \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar}\right)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n [(1 \otimes A(S_{t/\lambda^2}^\omega g_j)), H_I(t_1) \dots H_I(t_n)] \\ &= \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar}\right)^n \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \{[(1 \otimes A(S_{t/\lambda^2}^\omega g_j)), H_I(t_1)] H_I(t_2) \dots H_I(t_n) \\ &\quad + H_I(t_1) [(1 \otimes A(S_{t/\lambda^2}^\omega g_j)), H_I(t_2) \dots H_I(t_n)]\}.\end{aligned}$$

It can be shown that in the limit $\lambda \rightarrow 0$ only the commutator involving $H_I(t_1)$ contributes. Hence

$$\begin{aligned}\lim_{\lambda \rightarrow 0} \Xi_\lambda^b &= - \lim_{\lambda \rightarrow 0} \sum_{n=1}^{\infty} \left(\frac{\lambda}{i\hbar}\right)^{n-1} \int_0^{t/\lambda^2} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \\ &\quad \sum_{j,k} \langle S_{t/\lambda^2}^\omega g_j, S_{t_1/\lambda^2}^\omega g_k \rangle \text{Tr}\{|\phi\rangle\langle\psi| \otimes B_{T'/\lambda^2}^{(\omega,\lambda)\dagger}(f')|\Psi_R\rangle\langle\Psi_R| B_{T/\lambda^2}^{(\omega,\lambda)}(f) D_j^\dagger D_k H_I(t_2) \dots H_I(t_n)\},\end{aligned}$$

but this is the same as

$$\begin{aligned}&- \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^2} \sum_{j,k} \int_0^t ds \langle S_{t/\lambda^2}^\omega g_j, S_{s/\lambda^2}^\omega g_k \rangle D_j^\dagger D_k U_{s/\lambda^2}^{(\lambda)} \\ &= - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda^2} \sum_{j,k} \int_0^t ds \langle S_{t/\lambda^2}^\omega g_j, S_{s/\lambda^2}^\omega g_k \rangle \langle D_k^\dagger D_j \psi, G_\lambda(t) \rangle = - \sum_{j,k} (g_j|g_k)^{\omega-} \langle \psi, D_j^\dagger D_k G(t) \rangle,\end{aligned}$$

where we have used a technical lemma (6.3) of [1].

We now have

$$\begin{aligned}\langle \psi, G(t) \rangle &= \lim_{\lambda \rightarrow 0} \langle \psi, G_\lambda(t) \rangle = \langle \psi, G(0) \rangle + \lim_{\lambda \rightarrow 0} \int_0^t (\Gamma_\lambda(s) + \Xi_\lambda(s)) ds \\ &= \langle \psi, G(0) \rangle + \int_0^t ds \left\{ \sum_j \chi_{[S,T]}(s) (f|g_j)_\omega \langle D_j^\dagger \psi, G(s) \rangle \right. \\ &\quad \left. - \sum_j \chi_{[S',T']}(s) (g_j|f)^\omega \langle \psi, D_j G(s) \rangle - \sum_{j,k} (g_j|g_k)^{\omega-} \langle \psi, D_j^\dagger D_k G(s) \rangle \right\}.\end{aligned}$$

Here we have written $(g|f)^{\omega-}$ for $(g|f)_{Q+}^{\omega-}$ when $Q = 1$.

The quantum stochastic differential equation corresponding to this integral equation is

$$dU_t = \left\{ \sum_j (D_j \otimes dB_Q^{\omega\dagger}(g_j, t) - D_j^\dagger \otimes dB_Q^\omega(g_j, t)) - \sum_{j,k} (g_j|g_k)^{\omega-} D_j^\dagger D_k dt \right\} U_t.$$

The generalization to $Q > 1$ and several coupling frequencies ω is now obvious.

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